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SPECTRAL ANALYSIS OF THE FIBER SCHRÖDINGER OPERATORFOR A TWO-BOSON SYSTEM ON THE TWO-DIMENSIONAL LATTICE

Annotation. *We study the fiber Schrödinger operator for a two-boson system on the two-dimensional lattice with on-site, nearest-neighbor, and axial second nearest-neighbor interactions. At zero total quasi-momentum the operator is reduced to the coordinate-exchange-symmetric subspace, where the interaction has rank three. Using the Lippmann–Schwinger operator, the eigenvalue problem is reduced to a finite-dimensional Fredholm determinant. We derive the threshold asymptotics of this determinant, describe the corresponding critical sets in the interaction-parameter space, and prove that these sets divide the parameter space into regions on which the number of discrete eigenvalues is constant. This yields a complete classification of the number of eigenvalues below and above the essential spectrum.*

Keywords: *two-boson system, lattice Schrödinger operator, fiber operator, essential spectrum, Fredholm determinant, discrete spectrum.*

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IKKI O‘LCHAMLI PANJARADAGI IKKI BOZONLI SISTEMAGA MOS QAVAT SHRYODINGER OPERATORINING SPEKTRAL TAHLILI

Annotatsiya. *Biz ikki o‘lchamli panjaradagi, bir tugundagi, eng yaqin qo‘shni va o‘q yo‘nalishidagi ikkinchi qo‘shni o‘zaro ta’sirlarga ega ikki bozonli sistema uchun qavat Shrödinger operatorini o‘rganamiz. To‘la kvaziimpuls nol bo‘lganda operator koordinatalarni almashtirishga nisbatan simmetrik qism fazoga keltiriladi, bunda o‘zaro ta’sir operatorining rangi uchga teng. Lippmann–Shwinger operatori yordamida xos qiymatlar masalasi chekli o‘lchamli Fredholm determinantiga keltiriladi. Ushbu determinantning spektr chegaralaridagi asimptotikalarini topamiz, o‘zaro ta’sir parametrlari fazosidagi mos kritik to‘plamlarni tavsiflaymiz va bu to‘plamlar parametrlar fazosini diskret xos qiymatlar soni o‘zgarmas bo‘lgan sohalarga ajratishini isbotlaymiz. Natijada muhim spektrdan pastda va yuqorida joylashgan xos qiymatlar sonining to‘liq tasnifi olinadi.*

Kalit so'zlar: *ikki bozonli sistema, panjaradagi Shryodinger operatori, tolali operator, muhim spektr, Fredholm determinanti, diskret spektr.*

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СПЕКТРАЛЬНЫЙ АНАЛИЗ ВОЛОКОННОГО ОПЕРАТОРА ШРЁДИНГЕРА ДЛЯ СИСТЕМЫ ДВУХ БОЗОНОВ НА ДВУМЕРНОЙ РЕШЁТКЕ

Аннотация. *Исследуется волоконный оператор Шрёдингера для системы двух бозонов на двумерной решётке с одноузельным взаимодействием, взаимодействием ближайших соседей и осевым взаимодействием вторых соседей. При нулевом полном квазиимпульсе оператор редуцируется на подпространство, симметричное относительно перестановки координат, где оператор взаимодействия имеет ранг три. С помощью оператор Липпмана–Швингера задача на собственные значения сводится к конечномерному определителю Фредгольма. Получены пороговые асимптотики этого определителя, описаны соответствующие критические множества в пространстве параметров взаимодействия и доказано, что они разбивают пространство параметров на области, в которых число дискретных собственных значений постоянно. В результате получена полная классификация числа собственных значений ниже и выше существенного спектра.*

Ключевые слова: *система двух бозонов, оператор Шрёдингера на решётке, волоконный оператор, существенный спектр, определитель Фредгольма, дискретный спектр.*

1. Introduction. Discrete Schrödinger operators on lattices form a natural framework for the spectral analysis of few-particle quantum systems. Their discrete spectrum may contain bound states below and above the essential spectrum, and the number of such states depends on the interaction parameters [1,2]. Finite-range models with first- and second-neighbor interactions have been studied in [3–6,9].

We consider a two-boson fiber Schrödinger operator on the two-dimensional lattice with on-site, nearest-neighbor, and axial second nearest-neighbor interactions at zero total quasi-momentum. The Lippmann–Schwinger reduction yields a three-dimensional Fredholm determinant whose threshold asymptotics determine critical sets in parameter space and lead to a classification of the discrete eigenvalues below and above the essential spectrum.

2. The fiber Schrödinger operator. Let $\mathbb{T}^2 = (-\pi, \pi]^2$ be the two-dimensional torus equipped with the normalized Haar measure. For each $K \in \mathbb{T}^2$, consider the fiber Schrödinger operator

$$H_\mu(K) = H_0(K) + V_\mu$$

acts in $L^{2,\mathrm{e}}(\mathbb{T}^2)$, where

$$(H_0(K)f)(p) = \mathcal{E}_K(p)f(p),$$

and

$$\mathcal{E}_K(p) = \varepsilon(\frac{K}{2} + p) + \varepsilon(\frac{K}{2} - p) = 4\sum_{j=1}^{2}\left(1 - \cos\frac{K_j}{2}\cos p_j\right).$$

The interaction operator V_μ is given by

$$(V_\mu f)(p) = \mu_1 \int_{\mathbb{T}^2} f(q)\, dv(q) + 2\mu_2 \sum_{j=1}^{2} \cos p_j \int_{\mathbb{T}^2} \cos q_j\, f(q)\, dv(q)$$
$$+2\mu_3 \textstyle\sum_{j=1}^{2} \cos 2p_j \int_{\mathbb{T}^2} \cos 2q_j\, f(q)\, dv(q). \qquad (2.1)$$

2.1. The essential spectrum. The spectrum of the free fiber operator $H_0(K)$ is given by

$$\sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)], \quad K \in \mathbb{T}^2,$$

where

$$\begin{aligned} \mathcal{E}_{\min}(K) &= 4\textstyle\sum_{j=1}^{2}\left(1 - \cos\frac{K_j}{2}\right) \geq \mathcal{E}_{\min}(0) = 0, \\ \mathcal{E}_{\max}(K) &= 4\textstyle\sum_{j=1}^{2}\left(1 + \cos\frac{K_j}{2}\right) \leq \mathcal{E}_{\max}(0) = 16. \end{aligned} \qquad (2.2)$$

Since $V_\mu(K)$ is a finite-rank operator, by Weyl's theorem we have

$$\sigma_{\mathrm{ess}}(H_\mu(K)) = \sigma_{\mathrm{ess}}(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)].$$

Consequently,

$$\sigma_{\mathrm{disc}}(H_\mu(K)) \subset (-\infty, \mathcal{E}_{\min}(K)) \cup (\mathcal{E}_{\max}(K), \infty).$$

2.2. Symmetry decomposition at zero total quasi-momentum. We restrict to the case of zero total quasi-momentum. In this case, the dispersion relation takes the form

$$\mathcal{E}_0(p) = 4(2 - \cos p_1 - \cos p_2).$$

Both $H_0(0)$ and V_μ are invariant under the coordinate interchange

$$(p_1, p_2) \mapsto (p_2, p_1).$$

Hence, $H_\mu(0)$ admits the orthogonal decomposition

$$L^{2,\mathrm{e}}(\mathbb{T}^2) = L^{2,\mathrm{e,s}}(\mathbb{T}^2) \oplus L^{2,\mathrm{e,a}}(\mathbb{T}^2),$$

where

$$L^{2,\mathrm{e,s}}(\mathbb{T}^2) = \{f \in L^{2,\mathrm{e}}(\mathbb{T}^2) : f(p_1, p_2) = f(p_2, p_1)\},$$

and

$$L^{2,\mathrm{e,a}}(\mathbb{T}^2) = \{f \in L^{2,\mathrm{e}}(\mathbb{T}^2) : f(p_1, p_2) = -f(p_2, p_1)\}.$$

Both subspaces reduce $H_\mu(0)$, hence

$$H_\mu(0) = H_\mu^{\mathrm{e,s}}(0) \oplus H_\mu^{\mathrm{e,a}}(0). \tag{2.3}$$

$$\sigma\left(H_\mu(0)\right) = \sigma\left(H_\mu^{\mathrm{e,s}}(0)\right) \cup \sigma\left(H_\mu^{\mathrm{e,a}}(0)\right). \tag{2.4}$$

3. The Fredholm Determinant and Threshold Asymptotics. 3.1. The Lippmann–Schwinger operator.

We restrict our analysis to $H_\mu^{\mathrm{e,s}}(0)$ acting in $L^{2,\mathrm{e,s}}(\mathbb{T}^2)$. Introduce the orthonormal form factors

$$\phi_1(p) = 1, \qquad \phi_2(p) = \cos p_1 + \cos p_2, \qquad \phi_3(p) = \cos 2p_1 + \cos 2p_2.$$

Then

$$(V_\mu^{\mathrm{e,s}} f)(p) = \sum_{i=1}^{3} \mu_i \langle \phi_i, f \rangle \phi_i(p), \tag{3.1}$$

so that $rank V_\mu^{\mathrm{e,s}} \le 3$.

Accordingly,

$$H_\mu^{\mathrm{e,s}}(0) = H_0(0) + V_\mu^{\mathrm{e,s}},$$

where $H_0(0)$ denotes the restriction of the free fiber operator to $L^{2,\mathrm{e,s}}(\mathbb{T}^2)$. By (2.2),

$$\sigma_{\mathrm{ess}}\left(H_\mu^{\mathrm{e,s}}(0)\right) = [0{,}16]. \tag{3.2}$$

Let $n_-(H_\mu^{\mathrm{e,s}}(0))$ and $n_+(H_\mu^{\mathrm{e,s}}(0))$ denote the numbers of eigenvalues of $H_\mu^{\mathrm{e,s}}(0)$ in $(-\infty, 0)$ and $(16, +\infty)$, respectively. By the min–max principle,

$$n_-(H_\mu^{\mathrm{e,s}}(0)) + n_+(H_\mu^{\mathrm{e,s}}(0)) \le rank V_\mu^{\mathrm{e,s}} \le 3. \tag{3.3}$$

For $z \in \mathbb{R}\backslash[0{,}16]$, the resolvent of the free operator is defined by

$$R_0(0, z) = (H_0(0) - zI)^{-1}.$$

The Lippmann–Schwinger operator associated with $H_\mu^{\mathrm{e,s}}(0)$ is

$$L_\mu^{\mathrm{e,s}}(0, z) = -V_\mu^{\mathrm{e,s}} R_0(0, z). \tag{3.4}$$

Lemma 1. *Let $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$. Then $z \in \mathbb{R}\backslash[0{,}16]$ is an eigenvalue of $H_\mu^{\mathrm{e,s}}(0)$ with multiplicity m if and only if 1 is an eigenvalue of $L_\mu^{\mathrm{e,s}}(0, z)$ with the same multiplicity.*

Proof. This is the standard Lippmann–Schwinger principle; see [2].

For $z \in \mathbb{R}\backslash[0{,}16]$ define

$$a_{ij}(z) := \left\langle R_0(0, z)\phi_j, \phi_i \right\rangle = \int_{\mathbb{T}^2} \frac{\phi_i(p)\phi_j(p)}{\mathcal{E}_0(p) - z}\, d\nu(p), \quad i, j = 1{,}2{,}3. \tag{3.5}$$

Set

$$A(z) = (a_{ij}(z))_{i,j=1}^{3}, \qquad D_\mu = diag(\mu_1, \mu_2, \mu_3).$$

Since the range of $L_\mu^{\mathrm{e,s}}(0, z)$ is contained in $span\{\phi_1, \phi_2, \phi_3\}$, a nonzero solution of the Lippmann–Schwinger equation has the form

$$f = \sum_{j=1}^{3} x_j \phi_j.$$

Substitution into

$$f = L_\mu^{\mathrm{e,s}}(0, z) f$$

$$x_i + \mu_i \sum_{j=1}^{3} a_{ij}(z)x_j = 0, \qquad i = 1,2,3.$$

$$(I_3 + D_\mu A(z))x = 0, \quad x = (x_1, x_2, x_3)^{\mathrm{T}} \tag{3.6}$$

Hence a nontrivial solution exists if and only if

$$\det(I_3 + D_\mu A(z)) = 0.$$

Define the Fredholm determinant

$$\Delta_\mu^{e,s}(z) := \det(I_3 + D_\mu A(z)). \tag{3.7}$$

Lemma 2. *A point $z \in \mathbb{R}\backslash[0,16]$ is an eigenvalue of $H_\mu^{e,s}(0)$ if and only if*

$$\Delta_\mu^{e,s}(z) = 0.$$

Moreover, $\Delta_\mu^{e,s}(\cdot)$ has at most three zeros in $\mathbb{R}\backslash[0,16]$.

Proof. This follows directly from Lemma 1 and (3.6).

Lemma 3. *For all $i,j = 1,2,3$ and $z \in \mathbb{R}\backslash[0,16]$,*

$$a_{ij}(16 - z) = (-1)^{i+j+1} a_{ij}(z). \tag{3.8}$$

Proof. Using

$$\mathcal{E}_0(p + (\pi,\pi)) = 16 - \mathcal{E}_0(p), \qquad \phi_k(p + (\pi,\pi)) = (-1)^{k+1}\phi_k(p),$$

and the change of variables $p \mapsto \pi - p$ in (3.5) gives (3.8).

Proposition 1. *For $i,j = 1,2,3$, the functions $a_{ij}(z)$ are analytic and real-valued on $\mathbb{R}\backslash[0,16]$. Moreover, $a_{ii}(z)$ is positive and strictly increasing on $(-\infty, 0)$, and negative and strictly increasing on $(16, +\infty)$.*

As z approaches the spectral thresholds,

$$a_{ij}(z) = \begin{cases} -\frac{b_{ij}^-}{8\pi} ln\left(-\frac{z}{32}\right) + c_{ij}^- + o(1), & z \nearrow 0, \\ \frac{b_{ij}^+}{8\pi} ln\left(\frac{z-16}{32}\right) + c_{ij}^+ + o(1), & z \searrow 16. \end{cases} \tag{3.9}$$

Here

$$B^- = (b_{ij}^-)_{i,j=1}^3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}, \quad C^- = (c_{ij}^-)_{i,j=1}^3 = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{2-\pi}{\pi} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{4-2\pi}{\pi} \\ \frac{2-\pi}{\pi} & \frac{4-2\pi}{\pi} & \frac{\frac{88}{3}-10\pi}{\pi} \end{pmatrix}.$$

The upper-threshold coefficients are determined by

$$B^+ = SB^-S, \qquad C^+ = -SC^-S, \qquad S = diag(1,-1,1).$$

Proof. These properties follow by the argument of [9].

Lemma 4. *Let $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$. The function $\Delta_\mu^{e,s}(z)$ is analytic and real-valued on $\mathbb{R}\backslash[0,16]$. Moreover,*

$$\lim_{z\to-\infty} \Delta_\mu^{e,s}(z) = \lim_{z\to+\infty} \Delta_\mu^{e,s}(z) = 1.$$

At the spectral thresholds,

$$\Delta_\mu^{e,s}(z) = \begin{cases} -\frac{P_0^-(\mu)}{4\pi} ln\left(-\frac{z}{32}\right) + P_1^-(\mu) + o(1), & z \nearrow 0, \\ \frac{P_0^+(\mu)}{4\pi} ln\left(\frac{z-16}{32}\right) + P_1^+(\mu) + o(1), & z \searrow 16. \end{cases} \tag{3.10}$$

Here

$$P_0^\pm(\mu) = \frac{1}{2}\mu_1 + 2\mu_2 + 2\mu_3 \mp \frac{1}{4}\mu_1\mu_2 \mp \frac{32-9\pi}{3\pi}\mu_1\mu_3 \mp \frac{128-39\pi}{3\pi}\mu_2\mu_3 + \frac{128-39\pi}{24\pi}\mu_1\mu_2\mu_3, \tag{3.11}$$

and

$$P_1^\pm(\mu) = 1 \pm \frac{1}{2}\mu_2 \mp \frac{88-30\pi}{3\pi}\mu_3 - \frac{1}{16}\mu_1\mu_2 + \frac{4\pi-4-\pi^2}{\pi^2}\mu_1\mu_3 + \frac{3\pi^2+4\pi-48}{3\pi^2}\mu_2\mu_3 \mp \frac{3\pi^2+4\pi-48}{24\pi^2}\mu_1\mu_2\mu_3. \tag{3.12}$$

Proof. Analyticity follows from that of $\Delta_\mu^{e,s}(z)$ the matrix elements $a_{ij}(z)$. Since $A(z) \to 0$ as $z \to \pm\infty$,

$$\Delta_\mu^{e,s}(z) = \det(I_3 + D_\mu A(z)) \to 1.$$

Substitution of (3.9) into the determinant and Sylvester's identity give (3.10); direct calculation gives (3.11)-(3.12).

4. Geometry of the critical sets and main results. By Lemma 4, the lower and upper critical sets are determined by the vanishing of the leading logarithmic coefficients $P_0^-(\mu)$ and $P_0^+(\mu)$, respectively.

$$P_0^-(\mu) = 0 \text{ and } P_0^+(\mu) = 0.$$

Lemma 5. *The polynomials $P_0^\pm(\mu)$ can be represented as*

$$P_0^\pm(\mu) = (\mu_1 \mp 8)\left[\mu_2 Q_0^\pm(\mu_3) \mp R_0^\pm(\mu_3)\right] - 16Q_0^\pm(\mu_3), \tag{4.1}$$

where

$$Q_0^\pm(\mu_3) = \frac{128 - 39\pi}{24\pi}\mu_3 \mp \frac{1}{4}, \qquad R_0^\pm(\mu_3) = \frac{32 - 9\pi}{3\pi}\mu_3 \mp \frac{1}{2}.$$

The functions $Q_0^\pm$ and $R_0^\pm$ are affine in μ_3 and have the unique zeros

$$\mu_{3,0}^\pm = \pm\frac{6\pi}{128 - 39\pi}, \qquad \mu_{3,*}^\pm = \pm\frac{3\pi}{2(32 - 9\pi)},$$

respectively. In particular,

$$\mu_{3,0}^- < \mu_{3,*}^- < 0 < \mu_{3,*}^+ < \mu_{3,0}^+,$$

with

$$\mu_{3,0}^- \approx -3.441, \qquad \mu_{3,*}^- \approx -1.264, \qquad \mu_{3,*}^+ \approx 1.264, \qquad \mu_{3,0}^+ \approx 3.441.$$

Proof. Direct algebra from (3.11)-(3.12) gives (4.1); the remaining statements follow from the affine factors.

$$128 - 39\pi > 0, \quad 32 - 9\pi > 0, \qquad 128 - 39\pi < 4(32 - 9\pi).$$

Let

$$I_0^- = (\mu_{3,0}^-, +\infty), \quad I_1^- = (-\infty, \mu_{3,0}^-),$$

and

$$I_0^+ = (-\infty, \mu_{3,0}^+), \quad I_1^+ = (\mu_{3,0}^+, +\infty).$$

Lemma 6. *For each choice of sign, the set of points $(\mu_2, \mu_3) \in \mathbb{R}^2$ satisfying*

$$\mu_2 Q_0^\pm(\mu_3) \mp R_0^\pm(\mu_3) = 0 \tag{4.2}$$

is the graph of the function

$$\mu_2^\pm(\mu_3) = \pm\frac{R_0^\pm(\mu_3)}{Q_0^\pm(\mu_3)}, \qquad \mu_3 \in I_0^\pm \cup I_1^\pm. \tag{4.3}$$

This graph consists of two disjoint smooth unbounded connected curves

$$\tau_\alpha^\pm = \left\{(\mu_2, \mu_3) \in \mathbb{R}^2 : \mu_2 = \mu_2^\pm(\mu_3), \mu_3 \in I_\alpha^\pm\right\}, \quad \alpha = 0,1. \tag{4.4}$$

Moreover, the curves $\tau_0^\pm$ and $\tau_1^\pm$ divide the (μ_2, μ_3)-plane into three disjoint open unbounded connected components:

$$D_0^\pm := \left\{(\mu_2, \mu_3) \in \mathbb{R}^2 : \mu_3 \in I_0^\pm, \ \pm\mu_2 < \pm\mu_2^\pm(\mu_3)\right\}, \tag{4.5}$$

$$D_1^\pm := \left\{(\mu_2, \mu_3) \in \mathbb{R}^2 : \mu_3 \in I_0^\pm, \quad \pm\mu_2 > \pm\mu_2^\pm(\mu_3)\right\} \cup \left\{(\mu_2, \mu_3) \in \mathbb{R}^2 : \mu_3 = \mu_{3,0}^\pm\right\} \cup \left\{(\mu_2, \mu_3) \in \mathbb{R}^2 : \mu_3 \in I_1^\pm, \ \pm\mu_2 < \pm\mu_2^\pm(\mu_3)\right\}, \tag{4.6}$$

$$D_2^\pm := \left\{(\mu_2, \mu_3) \in \mathbb{R}^2 : \mu_3 \in I_1^\pm, \ \pm\mu_2 > \pm\mu_2^\pm(\mu_3)\right\}. \tag{4.7}$$

Proof. Solving (4.2) gives (4.3). Its pole yields the vertical asymptote, and the sign change across the two branches gives the components (4.5)-(4.7).

$$\mu_2 Q_0^\pm(\mu_3) \mp R_0^\pm(\mu_3) = 0, \quad Q_0^\pm(\mu_3) = 0$$

For each choice of sign, define

$$\mu_1^\pm(\mu_2, \mu_3) = \pm 8 + \frac{16Q_0^\pm(\mu_3)}{\mu_2 Q_0^\pm(\mu_3) \mp R_0^\pm(\mu_3)}. \tag{4.8}$$

Lemma 7. *For each choice of sign, the critical set*

$$\Gamma^{\pm} = \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : P_0^{\pm}(\mu) = 0\}$$

is the graph of the function $\mu_1^{\pm}(\mu_2, \mu_3)$ *defined in (4.8) on* $D_0^{\pm} \cup D_1^{\pm} \cup D_2^{\pm}$.

More precisely,

$$\Gamma^{\pm} = \Gamma_0^{\pm} \cup \Gamma_1^{\pm} \cup \Gamma_2^{\pm},$$

where

$$\Gamma_\alpha^{\pm} = \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : \mu_1 = \mu_1^{\pm}(\mu_2, \mu_3), (\mu_2, \mu_3) \in D_\alpha^{\pm}\}, \quad \alpha = 0,1,2. \tag{4.9}$$

The sets $\Gamma_0^{\pm}$, $\Gamma_1^{\pm}$, *and* $\Gamma_2^{\pm}$ *are pairwise disjoint smooth unbounded connected surfaces.*

To describe the connected components of their complement, set

$$\Upsilon_\alpha^{\pm} = \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in \tau_\alpha^{\pm}\}, \quad \alpha = 0,1. \tag{4.10}$$

Then

$$\mathbb{R}^3 \backslash \Gamma^{\pm} = C_0^{\pm} \cup C_1^{\pm} \cup C_2^{\pm} \cup C_3^{\pm},$$

where

$$C_0^{\pm} := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in D_0^{\pm},\ \pm\mu_1 < \pm\mu_1^{\pm}(\mu_2, \mu_3)\}, \tag{4.11}$$

$$C_1^{\pm} := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in D_0^{\pm},\ \pm\mu_1 > \pm\mu_1^{\pm}(\mu_2, \mu_3)\} \cup \Upsilon_0^{\pm}$$
$$\cup \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in D_1^{\pm},\ \pm\mu_1 < \pm\mu_1^{\pm}(\mu_2, \mu_3)\}, \tag{4.12}$$

$$C_2^{\pm} := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in D_1^{\pm},\ \pm\mu_1 > \pm\mu_1^{\pm}(\mu_2, \mu_3)\} \cup \Upsilon_1^{\pm}$$
$$\cup \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in D_2^{\pm},\ \pm\mu_1 < \pm\mu_1^{\pm}(\mu_2, \mu_3)\}, \tag{4.13}$$

$$C_3^{\pm} := \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 : (\mu_2, \mu_3) \in D_2^{\pm},\ \pm\mu_1 > \pm\mu_1^{\pm}(\mu_2, \mu_3)\}. \tag{4.14}$$

The sets $C_0^{\pm}, C_1^{\pm}, C_2^{\pm}, C_3^{\pm}$ *are disjoint unbounded connected components of* $\mathbb{R}^3 \backslash \Gamma^{\pm}$.

Proof. By Lemma 5 the critical equation is linear in the remaining parameter, so (4.8) defines a graph on each domain component. The inequalities (4.11)-(4.14) describe exactly the four connected components of the complement.

$$P_0^{\pm}(\mu) = (\mu_1 \mp 8)\left[\mu_2 Q_0^{\pm}(\mu_3) \mp R_0^{\pm}(\mu_3)\right] - 16 Q_0^{\pm}(\mu_3).$$

4.1. Discrete spectrum classification. We now determine the numbers of discrete eigenvalues of $H_\mu^{e,s}(0)$ below and above the essential spectrum.

Theorem 1. *For each choice of sign, the number of discrete eigenvalues of* $H_\mu^{e,s}(0)$ *at the corresponding side of the essential spectrum is determined as follows:*

1. *For* $\alpha \in \{0,1,2,3\}$,

$$n_{\pm}(H_\mu^{e,s}(0)) = \alpha, \qquad \mu \in C_\alpha^{\pm}.$$

2. *For* $\alpha \in \{0,1,2\}$,

$$n_{\pm}(H_\mu^{e,s}(0)) = \alpha, \qquad \mu \in \Gamma_\alpha^{\pm}.$$

Proof. Fix a sign. The operator depends continuously on the parameters, so the eigenvalue count is locally constant until an eigenvalue reaches the corresponding threshold. By Lemma 4 this can occur only on the critical set; hence the count is constant on each connected component.

$$\Gamma^{\pm} = \{\mu \in \mathbb{R}^3 : P_0^{\pm}(\mu) = 0\}.$$

The constant is found from a representative point of each component $C_\alpha^{\pm}$ by the min–max principle and (3.3), which gives

$$n_{\pm}(H_\mu^{e,s}(0)) = \alpha, \qquad \mu \in C_\alpha^{\pm}, \qquad \alpha = 0,1,2,3.$$

On a critical surface separating adjacent regions, variation of the corresponding coupling is a rank-one monotone perturbation; one eigenvalue reaches the threshold while the others remain outside the essential spectrum. Thus the count on the critical surface is one less than in the adjacent region with the larger count.

$$n_{\pm}(H_\mu^{e,s}(0)) = \alpha, \qquad \mu \in \Gamma_\alpha^{\pm}, \qquad \alpha = 0,1,2$$

Conclusion. For the two-boson fiber Schrödinger operator on the two-dimensional lattice with on-site, nearest-neighbor, and axial second nearest-neighbor interactions, we reduced the zero-quasi-

momentum spectral problem to a rank-three Fredholm determinant. Its threshold asymptotics yield the critical parameter sets and their geometric decomposition.

These sets divide parameter space into connected regions on which the number of discrete eigenvalues is constant. Combined with the min–max principle, this gives the complete classification of eigenvalues below and above the essential spectrum.

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