(10). Кудряшов Н.А., Чмыхов М.А. Приближенные решения SIR-модели для описания коронавируса. Вестник Национального исследовательского ядерного университета «МИФИ», 2020, 9(5), с. 404–11.

Urazbayev Gayrat Urazaliyevich (Professor, (DSc) of the Department of Mathematical Engineering Urgench State University; E-mail: gurazboev@gmail.com),
Babadjanova Aygul Komiljanovna (Senior Researcher, PhD Khorezm Branch of the V.I.Romanovskiy Institute of Mathematics, Academy of Sciences of the Republic of Uzbekistan; E-mail: aygul.babadjanova1987@gmail.com),
Eshchanova Gulchexra Sheribbayevna (Director of the Abu Rayhan Beruni Specialized School of Physics, Mathematics and Computer Science; E-mail: gulchehra050989@gmail.com)

MATRIX-TRIPLET CONSTRUCTION OF EXPLICIT SOLUTIONS TO THE NEGATIVE-ORDER KORTEWEG-DE VRIES EQUATION ON THE HALF-LINE WITH RATIONAL REFLECTION COEFFICIENTS

Annotation. *This paper considers the negative-order Korteweg–de Vries system on the halfline* $u_t = v_x,\ v_{xxx} - 4uv_x - 2u_xv + u_x = 0$, *and constructs explicit solutions by means of a matrix triplet method associated with the Marchenko integral equation. Starting from rational reflection coefficients and discrete spectral data, we construct a finite-dimensional triplet* $(A,\ B,\ C)$ *and derive a separable representation of the Marchenko kernel. This reduces the inverse problem to an algebraic matrix framework involving a matrix* $\Gamma(x;t)$. *Several explicit examples are presented, including purely continuous-spectrum solutions and solutions with discrete spectrum.*

Keywords: *negative-order KdV equation, half-line problem, inverse scattering transform, Marchenko equation, matrix triplet, explicit solutions.*

YARIM TO‘G‘RI CHIZIQDA RATSIONAL AKS ETTIRISH KOEFFITSIYENTLARIGA EGA MANFIY TARTIBLI KORTEVEG-DE FRIZ TENGLAMASINING ANIQ YECHIMLARINI MATRITSA-TRIPLETI KONSTRUKSIYASI ORQALI HOSIL QILISH

Annotatsiya. *Ushbu maqolada yarim o‘qda berilgan manfiy tartibli Korteveg–de Friz sistemasi* $u_t = v_x,\ v_{xxx} - 4uv_x - 2u_xv + u_x = 0$, *ko‘rib chiqilib, Marchenko integral tenglamasi bilan bog‘liq matritsaviy uchlik usuli yordamida aniq ye-chimlar quriladi. Ratsional akslanish koeffitsiyentlari va diskret spektral ma’lumotlardan boshlab, chekli o‘lchamli* $(A,\ B,\ C)$ *matritsaviy uchlik quriladi hamda Marchenko yadrosi* $\Gamma(x;t)$ *ning ajraluvchi ko‘rinishi hosil qilinadi. Natijada teskari masala matritsaviy tenglama ishtirokidagi algebraik matritsa sxemasiga keltiriladi. Maqolada uzluksiz spektrli ye-chimlar hamda diskret spektrga ega yechimlarni o‘z ichiga olgan bir nechta aniq misollar keltirilgan.*

Kalit so‘zlar: *manfiy tartibli KdF tenglamasi, yarim o‘q masalasi, teskari sochilish usuli, Marchenko tenglamasi, uchlik matritsa, aniq yechimlar.*

ПОСТРОЕНИЕ ЯВНЫХ РЕШЕНИЙ ОТРИЦАТЕЛЬНОГО УРАВНЕНИЯ КОРТЕВЕГА–ДЕ ФРИЗА НА ПОЛУПРЯМОЙ С РАЦИОНАЛЬНЫМИ КОЭФФИЦИЕНТАМИ ОТРАЖЕНИЯ МЕТОДОМ МАТРИЧНЫХ ТРИПЛЕТОВ

Аннотация. *В данной статье рассматривается система Кортевега-де Фриза отрицательного порядка на полупрямой* $u_t = v_x,\ v_{xxx} - 4uv_x - 2u_xv + u_x = 0$, *и строятся явные решения с помощью метода матричной тройки, связанного с интеграль-ным уравнением Марченко.Исходя из рациональных коэффициентов отражения и дискретных спектральных данных, строится конечномерная матричная тройка* $(A,\ B,\ C)$ *и выводится сепа-рабельное представление ядра Марченко* $\Gamma(x;t)$ *Это позволяет свести обратную задачу к алгеб-раической матричной схеме, содержащей матричное уравнение. Приведены несколько явных примеров, включая решения с чисто непрерывным спектром и решения с дискретным спектром.*

Ключевые слова: *уравнение КдФ отрицательного порядка, задача на полупрямой, метод обратного рассеяния, уравнение Марченко, матричный триплет, явные решения.*

MSC 2020: *35Q53, 37K15, 35Q51*

1 Introduction. The Korteweg-de Vries (KdV) equation is one of the fundamental nonlinear integrable evolution equations and plays a central role in soliton theory, inverse scattering, and nonlinear

wave propagation [3–7]. Beyond its classical form, the KdV hierarchy includes both positive and negative flows generated by the associated recursion operator. In particular, the negative-order members of this hierarchy have attracted considerable attention due to their rich integrable structure, connections with nonlocal reductions, and the presence of nonstandard wave phenomena such as soliton and kink-type solutions [8, 10, 11, 13].

The inverse scattering transform (IST) provides a powerful analytical framework for studying integrable equations on unbounded domains. In the context of the matrix negative-order Korteweg–de Vries equation in the commutative case, the IST was developed in [12] on the whole line. In that work, the time evolution of the scattering data associated with a matrix Sturm–Liouville operator with a self-adjoint potential was derived, leading to a general procedure for constructing solutions.

On the other hand, explicit solution constructions for KdV-type equations have also been extensively investigated. In particular, explicit solutions to the KdV equation in the first quadrant of the x-t plane, expressed in terms of elementary functions and associated with rational reflection coefficients, were obtained in [3]. In the reflectionless case, these solutions reduce to pure N-soliton solutions.The novelty of the present paper lies in the construction of explicit matrix-triplet solutions for the scalar negative-order KdV equation on the half-line, in the derivation of the logarithmic determinant representation under the induced time evolution of the scattering data, and in the positivity analysis of the resulting matrix $\Gamma(x;t)$ in explicit examples.

We consider the negative-order KdV system

$$u_t = v_x,\ v_{xxx} - 4uv_x - 2u_x v + u_x = 0, \tag{1.1}$$

on the half-line $x \in [0,\ \infty)$ and for $t \geq 0$.

In a similar spirit to [3], we begin with an initial profile $u_0(x)$ defined for $x \geq 0$ and extend it to the whole line by setting

$$u(x,\ 0) = \begin{cases} u_0(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

This extension is used only to define the associated scattering data and does not imply that the resulting solution satisfies homogeneous boundary conditions at $x = 0$ for $t > 0$. The function $u_0(x)$ introduced above serves as the initial profile for the reconstructed solution and is assumed to belong to the class of real-valued potentials associated with the one-dimensional half-line Schrödinger problem

$$-\psi_{xx} + u_0(x)\psi = k^2\psi, \tag{1.2}$$

subject to

$$\psi'(0) + ik\psi(0) = 0. \tag{1.3}$$

More precisely, we assume that

$$\int_0^\infty (1 + x)|u_0(x)|\, dx < \infty, \tag{1.4}$$

and that u_0 is sufficiently regular for the associated scattering data to exist.

In the present work, we further restrict to the case where the reflection coefficient is rational and the discrete spectrum is finite. For such scattering data, the corresponding potentials are smooth and exponentially decaying.

Under these assumptions, the associated scattering data

$$\left\{R(k),\ \{\kappa_j\}_{j=1}^N,\ \{c_j\}_{j=1}^N\right\}$$

are well defined and satisfy the usual analyticity, symmetry, and decay conditions, where $R(k)$ is the right reflection coefficient, $\kappa_j > 0$ are the discrete spectral parameters, and c_j are the associated norming constants. We refer to [1, 4–6, 9] for the standard reconstruction of such data from the half-line Schrödinger problem (1.2)–(1.3).

When the reflection coefficient is rational, as shown in [3], the Marchenko kernel admits a finite-dimensional separable representation and can be expressed in terms of a matrix triplet $(A,\ B,\ C)$:

$$\Omega(y;t)\ =\ Ce^{\frac{-tA^{-1}}{2}}\ e^{-yA}\mathrm{B}\,.$$

2 Construction of explicit solutions via matrix triplets. To construct explicit solutions, we describe the rational scattering data in a more explicit form. Let $\Pi R(k)$ denote the strictly proper rational part of the reflection coefficient corresponding to poles in the upper half-plane C^{+}. Because of the symmetry relation

$$R(-k^{*})\ =\ R(k)^{*},$$

the poles in C^{+} occur either on the imaginary axis or in pairs symmetric with respect to the imaginary axis. Accordingly, we use the partial fraction decomposition

$$\Pi R(k)=\sum_{j=1}^{n}\sum_{s=1}^{m_j}\left[\frac{(-i)^{s}\,(\epsilon_{js}\ +\ i\gamma_{js})}{(k\ -\ i\beta_j\ -\alpha_j\ )^{s}}+\frac{(-i)^{s}\ (\epsilon_{js}\ -\ i\gamma_{js})}{(k\ -\ i\beta_j\ +\alpha_j\ )^{s}}\right]+\sum_{j=2n+1}^{M}\sum_{s=1}^{mj}\frac{(-i)^{s}\ r_{js}}{(k\ -\ i\omega_j\ )^{s}}\qquad(2.1)$$

Here n is the number of symmetric pairs of complex poles, $M-2n$ is the number of purely imaginary poles, mj denotes the multiplicity of the corresponding pole, and all coefficients ϵ_{js}, γ_{js}, and r_{js} are real.

Following [18], (2.1) can be represented via the triplet $(A,\ B,\ C)$ such that

$$\Pi R(k)\ =\ -i\,C\ (kI\ -\ iA)^{-1}$$

The dimension of A is

$$J\ =\ N\ +\ 2\sum_{j=1}^{n}m_j\ +\ \sum_{j=2n+1}^{M}m_j\quad,$$

and the matrix A is block diagonal, while B and C are built blockwise from the pole and norming data. More precisely, the blocks corresponding to the pairs $\pm\alpha_j\ +\ i\beta_j$ are built from the $2\ \times\ 2$ matrices

$$\Lambda_j\ =\ \begin{bmatrix}\beta_j & -\alpha_j\\ \alpha_j & \beta_j\end{bmatrix},$$

the purely imaginary poles $i\omega_j$ generate Jordan blocks with eigenvalue ω_j, and each discrete eigenvalue κ_j contributes a $1\ \times\ 1$ block.

In [17], the inverse scattering transform for the matrix negative-order Korteweg–de Vries equation in the commutative case was studied, and the time evolution of the scattering data was obtained as

$$\kappa_j\ (t)\ =\kappa_j\ ,$$

$$c_j\ (t)\ =\ e^{\frac{-t}{2\kappa_j}}\ c_j\ ,\ 1\ \le\ j\ \le\ N$$

$$R(k,\ t)\ =\ e^{\frac{-t}{2\kappa_j}}\ R(k),\quad k\in R\ \setminus\ \{0\}.\qquad(2.2)$$

In the scalar $(1\ \times\ 1)$ reduction, the commutative matrix equation coincides with (1.1); accordingly, we adopt the same evolution law in the scalar setting. The time-dependent Marchenko equation reads

$$K(x,\ y;t)\ +\ \Omega(x\ +\ y;t)\ +\ \int_{x}^{\infty}K(x,\ z;t)\Omega(y\ +\ z;t)\ dz\ \ =\ 0,\ \ y\ >\ x\ >\ 0\,.\qquad(2.3)$$

The function Ω is determined by the time-evolved scattering data (2.2) through

$$\Omega(y;t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(k)e^{\frac{-t}{2\kappa}+iky}dk + \sum_{j=1}^{N} e^{\frac{-t}{2\kappa_j}-k_j y}. \tag{2.4}$$

Since $R(k)$ is rational, Ω can be written in separable form:

$$\Omega(y;t) = Ce^{\frac{-tA^{-1}}{2}} e^{-yA}B. \tag{2.5}$$

Hence

$$\Omega(x+y;t) = (Ce^{\frac{-tA^{-1}}{2}} e^{-xA})(e^{-yA}B). \tag{2.6}$$

Define

$$\Gamma(x;t) := I_J + \int_x^{\infty} e^{-zA}BC\, e^{-zA} e^{\frac{-tA^{-1}}{2}} dz \quad . \tag{2.7}$$

Since all eigenvalues of A satisfy $\Re\lambda > 0$, the matrix A is invertible. Moreover, the matrix exponential satisfies an estimate of the form

$$\| e^{-zA} \| \leq C(1 + z^m)e^{-cz},$$

for suitable constants C, $c > 0$ and some integer $m \geq 0$. This guarantees absolute convergence of the integral in (2.7) and justifies differentiation of $\Gamma(x;t)$ with respect to both x and t under the integral sign.

Lemma 1. Assume that $\Omega(y;t)$ is given by (2.5), and that $\Gamma(x;t)$ is invertible. Then a solution of (2.3) is

$$K(x, y;t) = -Ce^{\frac{-tA^{-1}}{2}} e^{-xA}\Gamma(x;t)^{-1} e^{-yA}B. \tag{2.8}$$

By the standard Marchenko reconstruction formula,

$$u(x, t) = -2\frac{\partial}{\partial x}K(x, x;t). \tag{2.9}$$

Hence

$$u(x, t) = 2\frac{\partial}{\partial x}\left[Ce^{\frac{-tA^{-1}}{2}} e^{-xA}\Gamma(x;t)^{-1} e^{-xA}B\right]. \tag{2.10}$$

Proposition 1. Assume that $\Gamma(x;t)$ is invertible. Then the potential (2.10) admits the representation

$$u(x, t) = -2\frac{\partial^2}{\partial x^2} log\, det\, \Gamma(x;t). \tag{2.11}$$

Proposition 2. Assume that

$$det\Gamma(x; 0) > 0,\ x \in [0, \infty),$$

and that

$$inf\ x \geq 0\ det\Gamma(x; 0) > 0.$$

Assume further that $\Gamma(x;t) \to \Gamma(x; 0)$ uniformly for $x \geq 0$ as $t \to 0$.

Then there exists $\tau > 0$ such that

$$det\Gamma(x;t) > 0,\ x \in [0, \infty),\ 0 \leq t < \tau.$$

Consequently, the corresponding solution is regular on $[0, \infty) \times [0, \tau)$.

3 Examples. In this section, we illustrate the construction with explicit choices of the triplet (A, B, C). Consider

$$\Pi R(k) = \frac{-2i\epsilon(k - i\beta) - 2\alpha\gamma}{(k - i\beta)^2 - \alpha^2}, \beta > 0, \tag{3.1}$$

with the associated matrices

$$A = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = 2\begin{bmatrix} \gamma & \epsilon \end{bmatrix}. \tag{3.2}$$

The eigenvalues of A A are $\beta \pm i\alpha$, and therefore $\Re\lambda > 0$. A direct computation shows that

$$\begin{aligned} det\Gamma(x;t) &= 1 - \frac{\alpha^2}{4\beta^2(\alpha^2 + \beta^2)}(\epsilon^2 + \gamma^2)e^{-4\beta x - \frac{\beta}{\alpha^2+\beta^2}t} \\ &+ \frac{1}{\alpha^2 + \beta^2} e^{-2\beta x - \frac{\beta}{2(\alpha^2+\beta^2)}t}\left[(\alpha\epsilon - \beta\gamma)\sin\left(\frac{\alpha}{2(\alpha^2 + \beta^2)}t - 2\alpha x\right)\right] \\ &+ (\alpha\gamma + \beta\epsilon)\cos\left(\frac{\alpha}{2(\alpha^2 + \beta^2)}t - 2\alpha x\right) \end{aligned} \tag{3.3}$$

Lemma 2. If

$$\sqrt{\epsilon^2 + \gamma^2} < \frac{2\beta\sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2} + \beta}, \tag{3.4}$$

then $det\Gamma(x;t) > 0$ for all $x \geq 0$ and $t \geq 0$.

Corollary 1. Under condition (3.4), the function $u(x, t) = -2\partial_x^2 log\ det\Gamma(x;t)$ defines a global regular solution of (1.1) on the quarter-plane.

Now let we consider

$$A = \begin{bmatrix} \beta & -\alpha & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & \kappa \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2\gamma & 2\epsilon & c \end{bmatrix}, \tag{3.5}$$

where $\beta > 0$ and $\kappa > 0$. For example, choosing $\alpha = \frac{\sqrt{3}}{2}$, $\beta = \frac{1}{2}$, $\kappa = 2$, $\epsilon = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $c = 3$, the profile of the resulting explicit regular solution can be visualized (Figure 1) in MATLAB.

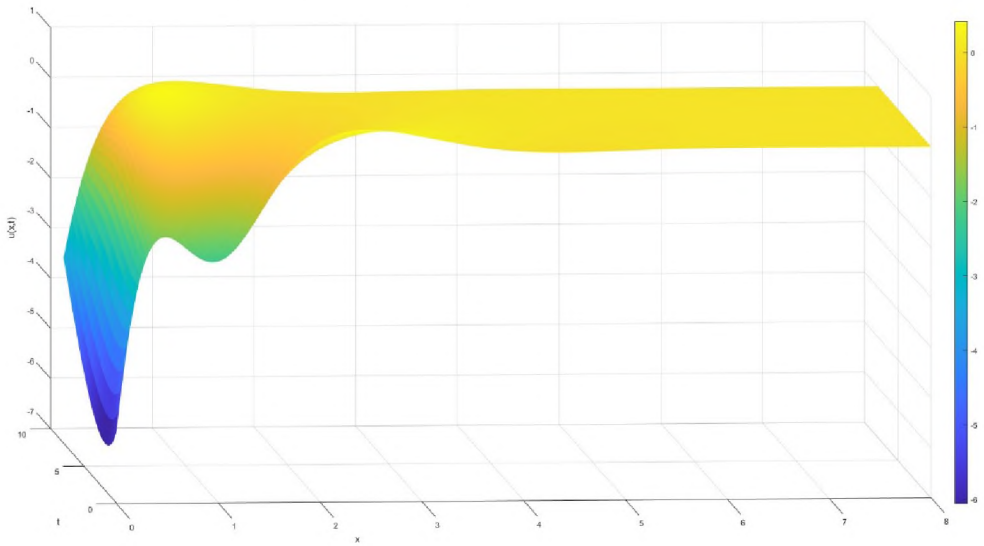


Figure 1: Surface plot of the solution $\mathbf{u(x, t)}$ corresponding to one discrete eigenvalue.

The presence of the discrete eigenvalue κ produces a localized soliton-type contribution, while the continuous-spectrum component produces an oscillatory dispersive background. Numerical evaluation over a large computational domain indicates that $det\Gamma(x;t) > 0$ for the above parameter values, which suggests global regularity for this example.

4. Conclusion. We have developed a matrix-triplet construction for explicit solutions of the negative-order KdV system on the half-line. For rational reflection coefficients and finitely many discrete eigenvalues, the Marchenko kernel admits a separable representation, which reduces the inverse problem to a finite-dimensional matrix framework. This leads to an explicit logarithmic determinant formula for the potential and provides a practical mechanism for constructing exact solutions.

References:

[1]. Aktosun T. & Klaus M. (2001). Inverse theory: problem on the line. In: Pike, E.R. & Sabatier, P. C. (eds.) Scattering, pp. 770–785, London, Academic Press.

[2]. Aktosun T., Klaus M. & van der Mee, C. (1992). Explicit Wiener–Hopf factorization for certain nonrational matrix functions. Integral Equations and Operator Theory, 15, pp. 879–900.

[3]. Aktosun T. & van der Mee C. (2006). Explicit solutions to the Korteweg-de Vries equation on the half-line. Inverse Problems, 22, pp. 2165–2174.

[4]. Deift P. & Trubowitz E. (1979). Inverse scattering on the line. Communications on Pure and Applied Mathematics, 32, pp. 121–251.

[5]. Faddeev L.D. (1967). Properties of the S-matrix of the one-dimensional Schrödinger equation. American Mathematical Society Translations, 65, pp. 139–166.

[6]. Fokas A.S. (2002). Integrable nonlinear evolution equations on the half-line. Communications in Mathematical Physics, 230, pp. 1–39.

[7]. Gardner C S., Greene J.M., Kruskal M.D. & Miura R.M. (1967). Method for solving the Korteweg-de Vries equation. Physical Review Letters, 19, pp. 1095–1097.

[8]. Kriecherbauer T., Urazboev G.U. & Babadjanova A.K. (2025). On the matrix negative-order Korteweg-de Vries equation – the commutative case. Wave Motion, 138, 103562.

[9]. Marchenko V.A. (1986). Sturm–Liouville Operators and Applications. Basel: Birkhäuser.

[10]. Qiao Z. & Li J. (2011). Negative-order KdV equation with both solitons and kink wave solutions. Europhysics Letters, 94, 50003.

[11]. Qiao Z. & Fan E. (2012). Negative-order Korteweg-de Vries equations. Physical Review E, 86, 016601.

[12] Urazboev G.U., Baltaeva I.I. & Ismoilov O.B. (2023). Integration of the negative-order Korteweg-de Vries equation by the inverse scattering method. Vestnik Udmurtskogo Universiteta. Matematika, Mekhanika, Komp'yuternye Nauki, 33, pp. 523–533.

[13]. Verosky J.M. (1991). Negative powers of Olver recursion operators. Journal of Mathematical Physics, 32, pp. 1733–1736.