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UNSTATIONARY FLOW OF MAXWELLIAN FLUID IN A CYLINDRICAL PIPE: ANALYTICAL SOLUTION

Annotation. *This study addresses the analytical problem of the unsteady flow of a rheologically complex fluid, specifically a Maxwellian fluid, within a cylindrical pipe under the influence of a constant pressure gradient. While stationary and oscillatory flows have been extensively studied, the transition processes in unsteady flows–where the fluid shifts from rest to a specific motion state–require deeper analytical exploration. This research aims to fill the scientific gap in theoretical models for non-stationary processes, which is vital for applications in biomedical engineering and hydraulic reliability.*

Keywords: *Maxwell's fluid, unsteady flow, cylindrical pipe, Laplace–Carson transform, Bessel functions, relaxation time, Poiseuille flow.*

SILINDRSIMON QUVURDA MAKSVEL SUYUQLIGINING NOSTATSIONAR OQIMI: ANALITIK YECHIM

Annotatsiya*. Ushbu tadqiqot reologik jihatdan murakkab suyuqlik, xususan, Maxwell fluidning doimiy bosim gradienti ta'sirida silindrsimon quvur ichidagi nostatsionar oqimining analitik masalasini ko'rib chiqadi. Statsionar va tebranishli oqimlar yetarlicha o'rganilgan bo'lsa-da, nostatsionar oqimlarda, ya'ni suyuqlikning tinch holatdan ma'lum harakat holatiga o'tish jarayonlari chuqurroq analitik tahlilni talab etadi. Mazkur tadqiqot nostatsionar jarayonlar uchun nazariy modellar sohasidagi ilmiy bo'shliqni to'ldirishga qaratilgan bo'lib, bu yo'nalish biotibbiyot muhandisligi hamda gidravlik tizimlar ishonchliligi uchun muhim ahamiyat kasb etadi.*

Kalit so'zlar*: Maksvel suyuqligi, nostatsionar oqim, silindrsimon quvur, Laplace–Carson transform, Bessel functions, relaksatsiya vaqti, Poiseuille flow.*

НЕСТАЦИОНАРНОЕ ТЕЧЕНИЕ МАКСВЕЛЛОВСКОЙ ЖИДКОСТИ В ЦИЛИНДРИЧЕСКОЙ ТРУБЕ: АНАЛИТИЧЕСКОЕ РЕШЕНИЕ

Аннотация*. В данной работе рассматривается аналитическая задача нестационарного течения реологически сложной жидкости, в частности, Maxwell fluid – в цилиндрической трубе под действием постоянного градиента давления. Несмотря на то, что стационарные и колебательные течения были подробно исследованы, переходные процессы в нестационарных режимах, когда жидкость переходит из состояния покоя к определённому режиму движения, требуют более глубокого аналитического анализа. Настоящее исследование направлено на восполнение научного пробела в области теоретических моделей нестационарных процессов, что имеет важное значение для биомедицинской инженерии и обеспечения надёжности гидравлических систем.*

Ключевые слова*: максвелловская жидкость, нестационарное течение, цилиндрическая труба, Laplace–Carson transform, Bessel functions, время релаксации, Poiseuille flow.*

Introduction. The study of unsteady fluid flows is a fundamental aspect of modern fluid mechanics,

particularly when dealing with transitions between different states of motion. Unlike stationary flows, which remain constant over time, or oscillatory flows, which follow periodic functions, unsteady flows involve complex temporal changes that occur during the transition of a fluid from one state to another. Understanding these transitions is crucial for ensuring the reliability and efficiency of various hydraulic and technical systems. Historically, significant research has been conducted on the unsteady flow of Newtonian fluids in channels and pipes. Early experimental work, such as that by Richardson (1929), revealed that velocity profiles in oscillatory motion differ fundamentally from the classical stationary Poiseuille flow. Later, Womersley (1955) applied these concepts to biomechanics, modeling blood as a Newtonian fluid to study the circulatory system. Today, modeling blood and other biological fluids as viscous media is essential for medical diagnostics and the optimization of drug delivery systems. However, many real-world fluids, particularly in biomedical engineering and microfluidics, exhibit rheologically complex behavior that cannot be fully described by the Newtonian model. In devices such as microfluidic chips and pneumatic micropumps, hydrodynamic processes are often interpreted as laminar pulsating flows where the presence of an elasticity component significantly alters flow dynamics. Despite their importance, theoretical and mathematical models for non-stationary processes in such systems remain insufficiently developed, with existing research often limited to empirical results. The Maxwell model is one of the most effective frameworks for capturing the dual viscous and elastic nature of such fluids. The inclusion of a relaxation parameter in this model ensures that transient characteristics are qualitatively different from classical solutions.

Literature review. This paper aims to bridge the existing scientific gap by providing a detailed analytical solution for the unsteady flow of a Maxwellian fluid in a cylindrical pipe under a constant pressure gradient. Using the Laplace–Carson transformation, we investigate the influence of relaxation processes on velocity distribution, providing a theoretical foundation for the analysis of rheologically complex media in transition states. It can be noted that many scientific and practical studies have been conducted by domestic and foreign scientists on the issues of oscillatory and unsteady flows of Newtonian fluids in channels and pipes. In particular, in the research works [1,2], the unsteady and stationary oscillatory flows of viscous fluids in channels and pipes were sufficiently studied. For the first time, the characteristic features of the velocity profiles in the oscillatory flow of viscous fluids in cylindrical pipes were presented in the form of experimental results in the work [3]. In particular, it was found that the velocity distribution profile in the oscillatory flow is fundamentally different from the velocity distribution profile in the stationary Poiseuille flow. Later, such studies were carried out by Womersley [4] in his scientific research works on the oscillatory flow of viscous fluids, considering blood as a Newtonian fluid and applying it to the circulatory system of biomechanics.

The study of the dynamics of laminar, unsteady and oscillatory motion of blood by modeling it as a viscous fluid is a crucial methodological factor in the diagnosis of the vital activity of the human body from a mechanical point of view and in the optimization of drug delivery systems [5-6]. The practical significance of these studies is especially evident in improving the efficiency of microfluidic chips and pneumatic micropumps used in biomedical engineering [6,7]. The hydrodynamic processes in these devices are mainly interpreted as laminar pulsating flows in prismatic microchannels with a rectangular cross section.

In this section, a mathematical model describing the unsteady flow of a rheologically complex fluid in a cylindrical pipe, obeying the Maxwell model, is formulated. The following geometric and hydrodynamic assumptions are made in modeling the problem:

Coordinate system: Using a cylindrical coordinate system, z we direct the axis along the pipe axis, and the radius r We define it through.

Geometric dimensions: Pipe radius R The ratio of the pipe length to its radius is large enough $(L\ ?\ R)$ Since the edge effects at the inlet and outlet of the flow and the transverse components of the velocity $(v_r = 0,\ v_\theta = 0)$ is not taken into account.

Flow characteristics: The flow is considered one-dimensional and unsteady, and the velocity field depends only on the radial coordinate and time: $v = v(r,t)$ Under these conditions, the system of differential equations describing the unsteady motion of a rheologically complex fluid in a cylindrical pipe is expressed in a simplified form as follows:

$$\begin{cases} \rho\dfrac{\partial v_x}{\partial t} = -\dfrac{\partial p}{\partial x} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau), \ \tau = \alpha_1\tau_1^0 + \alpha_2\tau_2^0, \\ \tau_1^0 = \mu_1^0\dfrac{\partial v_x}{\partial r}, \ \theta\dfrac{\partial \tau_2^0}{\partial t} + \tau_2^0 = \mu_2^0\dfrac{\partial v_x}{\partial r}, \ \mu = \alpha_1\mu_1^0 + \alpha_2\mu_2^0 \end{cases} . \quad (1)$$

Here:

ρ – liquid density;

p – hydrodynamic pressure;

τ – test voltage;

θ – relaxation time of the liquid;

τ_1^0, μ_1^0 – Newtonian fluid tensile stress and dynamic viscosity coefficient;

τ_2^0, μ_2^0 – Maxwell fluid shear stress and dynamic viscosity coefficient;

t – time;

r – radial coordinate.

To solve this system of equations (1), it is necessary to formulate initial and boundary conditions. The initial time $t = 0$ At the moment the fluid is considered to be at rest throughout the entire pipe section [8, 9]. In this case, the velocity and shear stresses are assumed to be zero:

$$t = 0 \quad \partial a \quad v_x = 0, \quad \frac{\partial p}{\partial x} = 0. \quad (2)$$

At positive values of time, the motion occurs under the influence of a constant pressure gradient, the boundary conditions for this case are defined as follows [10,11]:

- Viscosity condition: On the pipe wall. $(r = R)$ The fluid particles are stationary, meaning the velocity is zero.
- Symmetry condition: On the pipe axis $(r = 0)$ the velocity profile has an extremum, i.e. the velocity gradient is zero.

$$v(R,t) = 0, \quad \left.\frac{\partial v}{\partial r}\right|_{r=0} = 0 \quad (3)$$

The system of equations (1) formed, together with the initial and boundary conditions (2) and (3), fully describes the unsteady hydrodynamic behavior of a rheologically complex fluid in a cylindrical pipe [12,13].

Research Methodology. To solve the system of partial differential equations (1) with the given initial and boundary conditions (2) and (3), we use the Laplace-Carson integral substitution. This method allows us to reduce the differential equation to an operator form by transferring the time-dependent functions to the domain of representations. Time variable (t) According to the definition of the Laplace-Carson transform on , it is optional $f(t)$ function description $\overline{f}(p)$ is defined by the following integral:

$$\overline{f}(p) = p\int_0^\infty e^{-pt} f(t)dt$$

To apply this substitution to system (3.1) and conditions (2)-(3), we use the following properties:

For the derivative of a function: $L\left\{\dfrac{\partial f}{\partial t}\right\} = p\overline{f(p)} - pf(0)$

Considering the initial conditions: $t = 0$ at $v = 0$ and $\tau = 0$ Since , the initial values in the operator representation are zero.

As a result of the substitution (1), the system of differential equations with partial derivatives is obtained in radial coordinates r It turns into an ordinary differential equation with respect to . For this, we introduce the following integrals:

$$\bar{v}_x = p\int_0^\infty e^{-pt} v_x dt, \qquad \bar{\tau}_{rx} = p\int_0^\infty e^{-pt} \tau_{rx} dt. \tag{4}$$

Here p – replacement parameter.

Based on the Laplace-Carson transformations, we use these relations to represent the quantities in equations (1) and the initial and boundary conditions in (2), (3):

$$\begin{aligned} &\frac{\partial v_x}{\partial t} \to p\bar{v}_x, \qquad \frac{\partial v_x}{\partial r} \to \frac{d\bar{v}_x}{dr} \\ &\frac{\partial \tau_{rx}}{\partial t} \to p\bar{\tau}_{rx}, \qquad \frac{\partial p}{\partial x} \to \frac{d\bar{p}}{dx}. \end{aligned} \tag{5}$$

With these substitutions, the system of equations (3.1) takes the following form in the figure:

$$\begin{cases} \dfrac{d\bar{\tau}}{dr} + \dfrac{1}{r}\bar{\tau} - \rho p\bar{v}_x = \dfrac{d\bar{p}}{dx}, \ \tau = \alpha_1\tau_1^0 + \alpha_2\tau_2^0, \\ \tau_1^0 = \mu_1^0 \dfrac{\partial \vartheta_x}{\partial r}, \ \theta\dfrac{\partial \tau_2^0}{\partial t} + \tau_2^0 = \mu_2^0 \dfrac{\partial v_x}{\partial r}, \ \mu = \alpha_1\mu_1^0 + \alpha_2\mu_2^0 \end{cases}. \tag{6}$$

Now from the second, third and fourth equations of the system of equations (6) $\bar{\tau}_{rx}$ we find:

$$\bar{\tau}(r) = m\left(\frac{a_1\mu_1^0}{\mu} + \frac{a_2\mu_2^0}{\mu}\frac{1}{1+p\theta}\right)\frac{\partial v_x}{\partial r} = \mu\mu^*(p)\frac{\partial v_x}{\partial r} \tag{7}$$

Here $\bar{\mu}(p) = \mu\left(\dfrac{\alpha_1\mu_1^0}{\mu} + \dfrac{\alpha_2\mu_2^0}{\mu}\dfrac{1}{1+p\theta}\right)$.

Substituting the defined expression (3.7) into equation (3.6), we get

$$\frac{d^2\bar{v}_x}{dr^2} + \frac{1}{r}\frac{d\bar{v}_x}{dr} - \frac{\rho p}{\bar{\mu}(p)}\bar{v}_x = \frac{1}{\bar{\mu}(p)}\frac{d\bar{p}}{dx} \tag{8}$$

We form equation (8). The resulting equation (8) is a second-order homogeneous ordinary differential equation. To solve this equation, first its homogeneous part is solved, and then its heterogeneous part is solved. Therefore, we look for the solution of equation (8) as two solutions in the following form:

$$\bar{v}_x = \bar{v}_x^{-1} + \bar{v}_x^*.$$

Then the homogeneous part of equation (8) has the following form,

$$\frac{d^2\bar{v}_x^{-1}}{dr^2} + \frac{1}{r}\frac{d\bar{v}^{-1}}{dr} - \frac{\rho p}{\bar{\mu}(p)}\bar{v}^{-1} = 0$$

its fundamental solutions consist of the following Bessel functions:

$$I_0(\kappa r), \quad K_0(\kappa r).$$

Solution of the non-homogeneous part $\bar{v}_x^* = A$ We are looking for a constant quantity. Substituting this quantity into equation (8), we obtain the following equation: $-\dfrac{\rho p}{\bar{\mu}(p)}A = \dfrac{1}{\bar{\mu}(p)}\dfrac{d\bar{p}}{dx}$. And from this A We determine the size: $A = -\dfrac{1}{\rho p}\dfrac{d\bar{p}}{dx}$.

Using these solutions, we determine the general solution of equation (8) in the following form:

$$\bar{v}_x(r) = C_1 I_0(\kappa r) + C_2 K_0(\kappa r) - \frac{1}{\rho p}\frac{d\bar{p}}{dx}. \tag{9}$$

Here $K_0(\kappa r)$ function $r = 0$ tends to infinity when . Therefore, to obtain a finite solution, $C_2 = 0$ must be. Then the solution to (9) will be as follows:

$$\bar{v}_x(r) = C_1 I_0(\kappa r) - \frac{1}{\rho p}\frac{d\bar{p}}{dx}. \tag{10}$$

C_1 To find the value of the unknown coefficient, we use the condition that the fluid movement on the pipe wall is zero, i.e. $r = R$ when $v = 0$ will be:

$$0 = C_1 I_0(\kappa R) - \frac{1}{\rho p}\frac{d\bar{p}}{dx}.$$

From this C_1 we find the unknown coefficient:

$$C_1 = \frac{\frac{1}{\rho p}\frac{d\bar{p}}{dx}}{I_0(\kappa R)}.$$

By substituting this value into the solution (10), the following resulting solution is obtained:

$$\bar{v}_x(r) = \frac{1}{\rho p}\left(-\frac{d\bar{p}}{dx}\right)\left[1 - \frac{I_0(\kappa r)}{I_0(\kappa R)}\right]. \tag{11}$$

Here— κ a parameter that describes the rheological and inertial properties of a medium

$$\kappa = \sqrt{\frac{\rho p}{\bar{\mu}(p)}}.$$

The abstract argument Bessel function in the solution of (11) can be expressed by this relation $I_n(x) = (-i)^n J_n(ix)$ We substitute it into a Bessel function with real arguments and obtain the following solution:

$$\bar{v}_x(r) = \frac{1}{\rho}\frac{1}{p}\left(-\frac{dp}{dx}\right)\left(1 - \frac{J_0(i\kappa r)}{J_0(i\kappa R)}\right). \tag{12}$$

The resulting solution (12) is the solution of equation (6) in the image, and to find its real solution, we use the Laplace-Carson transformation formula:

$$v_x(r,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}\frac{1}{\rho p}\left(-\frac{dp}{dx}\right)\left(1 - \frac{J_0(i\kappa r)}{J_0(i\kappa R)}\right)\frac{dp}{p}. \tag{13}$$

To calculate the integral of (13) with a complex variable, we use the fact that this integral is a Laplace integral and, taking into account that the function under the integral is a meromorphic function, we determine its special points.

These special points are:

$$p = 0, \quad s = -v\frac{\bar{p}_{1,n}}{R^2}, \quad p = -v\frac{\bar{p}_{2,n}}{R^2}.$$

Here $\bar{p}_{1,n}$, $\bar{p}_{2,n}$ values $J_0(i\kappa R) = 0$ solutions of the transcendental equation.

Therefore, we find the value of the integral (13) using the theory of residues. The special points at these values consist only of simple poles. Therefore, we use the expansion of the meromorphic function into a simple fraction. To do this, we write the expression under the integral in fractional form in this form:

$$\frac{F_1(p)}{F_2(p)} = \frac{1}{\rho p}\left(-\frac{dp}{dx}\right)\frac{\left(J_0(i\kappa R) - J_0(i\kappa r)\right)}{J_0(i\kappa R)}\frac{e^{pt}}{p} = \frac{C_0}{p} + \sum_{k=1}^{2}\frac{C_{kn}}{p - p_{kn}}. \tag{14}$$

$$F_1(p)=\left(-\frac{dp}{dx}\right)\left(J_0\left(i\kappa R\right)-J_0\left(i\kappa r\right)\right)e^{pt};\quad F_2(p)=\rho p^2 J_0\left(i\kappa R\right)$$

Here C_0 To find the remainder, multiply both sides of equation (14) p We multiply by and p We calculate this limit by trying to zero:

$$C_0=\lim_{s\to 0}\frac{pF_1(p)}{F_2(p)}=\lim_{s\to 0}\frac{p\left(-\frac{dp}{dx}\right)\left(J_0\left(i\kappa R\right)-J_0\left(i\kappa r\right)\right)e^{pt}}{\rho p^2 J_0\left(i\kappa R\right)}=\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2\left(1-\frac{r^2}{R^2}\right). \quad (15)$$

When calculating Bessel functions, its $J_0(x)=1-\left(\frac{x}{2}\right)^2+\cdots$ used to spread out in the visible row.

Now C_{in} To find (14), multiply both sides of equation (14) by $p-p_{in}$ multiplying by the expression, p what p_{in} We strive for the expression and calculate the limit:

$$C_{kn}=\lim_{p\to p_{kn}}\frac{F_1(p)}{\frac{F_2(p)-F_2(p_{kn})}{p-p_{kn}}}=\frac{F_1(p_{kn})}{F_2'(p_{kn})}. \quad (16)$$

Here p_{in} value $J_0\left(i\kappa R\right)=0$ is found by finding a solution to the transcendental equation. It is known that, $\beta_n=i\kappa R$ and in this β_n $J_0\left(\beta_n\right)=0$ was the solution of the equation. Now $\beta_n=i\kappa R$ from the equation p We find the values of , where $\nu=\frac{\mu}{\rho}$, we can say, p new parameter $p=-\frac{\nu}{R^2}\overline{p}$ We determine it through. In that case $\overline{p}$ is determined from this equation:

$$\left(-\overline{p}\right)+\beta_n^2\mu^*\left(\overline{p}\right)=0, \quad (17)$$

here β_n $-J_0\left(x\right)=0$ the roots of the equation.

Now

$$\mu^*(s)=\left(\frac{\alpha_1\mu_1^0}{\mu}+\frac{\alpha_2\mu_2^0}{\mu}\frac{1}{1+p\theta}\right) \quad (18)$$

Using the fact that , we obtain the following transcendental equation:

$$\overline{p}=\beta_n{}^2\left(\frac{\alpha_1\mu_1^0}{\mu}+\frac{\alpha_2\mu_2^0}{\mu}\frac{1}{1-\overline{p}De}\right),$$

here $De=\frac{\nu\theta}{R^2}$.

From this equation, this quadratic equation follows:

$$\overline{p}^2De-(1+\overline{p}X\beta_n^2De)\overline{p}+\beta_n^2=0. \quad (19)$$

There are two solutions to this equation, which are $\overline{p}_{1n,}$ $\overline{p}_{2n}$ is defined in appearance as follows:

$$\overline{p}_{1,2}=\frac{1+X\beta_n^2De\pm\sqrt{1+2\beta_n^2De(X-2)+X^2\beta_n^4De^2}}{2De}. \quad (20)$$

Now $C_{kn}=\lim_{p\to p_{in}}\frac{F_1(p)}{\frac{F_2(p)-F_2(p_{kn})}{p-p_{kn}}}=\frac{F_1(p_{kn})}{F_2'(p_{kn})}$ we find the coefficient. To do this, first $F_1(p_{kn})$

and then $F_2'(p_{kn})$ It will be necessary to calculate .

It is known that $F_1(p)=\left(-\frac{dp}{dx}\right)\left(J_0\left(i\kappa R\right)-J_0\left(i\kappa r\right)\right)e^{pt}$, from this $p=-\frac{\nu}{R^2}\overline{p}_{kn}$ is, $\overline{p}_{kn}$ for $F_1(\overline{p}_{kn})$ will be equal to:

$$F_1(\overline{p}_{in})=\sum_{n=1}^{\infty}\sum_{k=1}^{2}\left(-\frac{dp}{dx}\right)\left(-J_0\left(\beta_n\frac{r}{R}\right)\right)e^{-\frac{\nu}{R^2}\overline{p}_{kn}t}.$$

Now $F_2'(p_{kn})$ We calculate . $F_2(p)=\rho p^2 J_0\left(i\kappa R\right)$ It is known that this expression κ We find the derivative of:

$$F_2'(p)=2\rho pJ_0\left(i\kappa R\right)+\rho p^2J_0'\left(i\sqrt{\frac{\rho p}{\mu_k(p)}}R\right)=2\rho pJ_0\left(\beta_n\right)+\rho p^2J_0'\left(i\kappa R\right)=$$

$$=\rho p^2J_0'\left(i\kappa R\right)=-\rho p^2J_1\left(\beta_n\right)\left(i\kappa R\right)_s'=-\rho p^2J_1\left(\beta_n\right)\left(\frac{iR}{\sqrt{\nu}}\sqrt{\frac{p}{\mu^*(p)}}\right)'.$$

Here $J_0'(x)=-J_1(x)$ It was noted that. $\left(i\kappa R\right)_p'$ The derivation of the expression was found in the previous paragraph. Using this, we derive this, here $\left(i\kappa R\right)=\beta_n$:

$$\left(\sqrt{\frac{p}{\mu^*(p)}}\right)_p'=\frac{1}{2\left(\sqrt{\frac{p}{\mu^*(p)}}\right)}\left(\frac{p}{\mu^*(p)}\right)_p'=\frac{1}{2\left(\sqrt{\frac{p}{\mu^*(p)}}\right)}\frac{\mu^*(p)-p\left(\mu^*(p)\right)_s'}{\mu^{*2}(p)}.$$

$i\kappa R=\beta_n$ or $\frac{iR}{\sqrt{\nu}}\sqrt{\frac{p}{\mu^*(p)}}=\beta_n$ was. From this $\sqrt{\frac{p}{\mu^*(p)}}=\beta_n\frac{\sqrt{\nu}}{iR}$, $\mu^*(p)=-\frac{R^2p}{\nu\beta_n^2}$ we find that is. Then

$$\frac{iR}{\sqrt{\nu}}\left(\sqrt{\frac{p}{\mu^*(p)}}\right)_p'=\frac{iR}{\sqrt{\nu}}\frac{iR}{2\beta_n\sqrt{\nu}}\frac{\nu^2\beta_n^4}{R^4p^2}\left(\mu^*(p)-p\left(\mu^*(p)\right)_p'\right)=$$

$$=-\frac{\nu\beta_n^3}{2R^2p^2}\left(\mu^*(p)-p\left(\mu^*(p)\right)_p'\right).$$

Taking this into account, we get:

$$F_2'(p)=-\rho p^2J_1\left(\beta_n\right)\left(\frac{iR}{\sqrt{\nu}}\sqrt{\frac{p}{\mu*(p)}}\right)'=\rho p^2J_1\left(\beta_n\right)\frac{\nu\beta_n^3}{2R^2p^2}\left(\mu^*(p)-p\left(\mu^*(p)\right)_p'\right)=$$

$$=\rho J_1\left(\beta_n\right)\frac{\nu\beta_n^3}{2R^2}\left(\mu^*(p)-p\left(\mu^*(p)\right)_p'\right).$$

Here $\mu^*(p)=\left(\frac{\alpha_1\mu_1^0}{\mu}+\frac{\alpha_2\mu_2^0}{\mu}\frac{1}{1+p\theta}\right)$ and $\mu^{*'}\left(\overline{p}\right)=-\frac{\alpha_2\mu_2^0}{\mu}\frac{\theta}{(1+p\theta)^2}$ considering that, then

$$\mu^*(p)=X+Z\frac{1}{1+p\theta};\ \mu^{*\prime}(p)=\frac{-\theta Z}{(1+p\theta)^2},$$

$$\mu^*(p)-p\mu^{*\prime}(p)=X+Z\frac{1}{1+p\theta}-\frac{-p\theta Z}{(1+p\theta)^2}=\frac{1+2\theta p+\theta^2p^2X}{(1+p\theta)^2}$$

is formed. And from this

$$F_2'(\overline{p})=\rho J_1(\beta_n)\frac{\nu\beta_n^3}{2R^2}\left(\mu^*(\overline{p})-\overline{p}\left(\mu_k^*(\overline{p})\right)_{\overline{p}}'\right)=\rho J_1(\beta_n)\frac{\nu\beta_n^3}{2R^2}\frac{1+2\theta p+\theta^2p^2X}{(1+p\theta)^2}$$

It follows that this is the value found. $\overline{p}$ of p_{kn} If we sum them by their values, then $F_2'(\overline{p})$ This value is generated for:

$$F_2'(\overline{p}_{kn})=\sum_{n=1}^{\infty}\sum_{k=1}^{2}\rho J_1(\beta_n)\frac{\nu\beta_n^3}{2R^2}\frac{1-2De\overline{p}_{kn}+De^2\overline{p}_{kn}{}^2X}{(1-De\overline{p}_{kn})^2}.$$

Found $F_1(\overline{p})$ and $F_2'(\overline{p})$ values $C_{kn}=\frac{F_1(p_{kn})}{F_2'(p_{kn})}$ putting it in equality, $C_{k,in}$ we find:

$$C_{kn}=\frac{F_1(p_{kn})}{F_2'(p_{kn})}=\sum_{n=1}^{\infty}\sum_{k=1}^{2}\frac{2R^2\left(-\frac{dp}{dx}\right)\left(-J_0\left(\beta_n\frac{r}{R}\right)\right)e^{-\frac{\nu}{R^2}\overline{p}_{kn}t}}{J_1(\beta_n)\mu\beta_n^3\frac{1-2De\overline{p}_{kn}+De^2\overline{p}_{kn}{}^2X_1}{(1-De\overline{p}_{kn})^2}}.$$

C_0 and C_{kn} Considering the values of the coefficients, the solution to the equation is determined as follows:

$$v_x(r,t)=\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2\left(1-\frac{r^2}{R^2}-8\sum_{n=1}^{\infty}\sum_{k=1}^{2}\frac{J_0\left(\beta_n\frac{r}{R}\right)e^{-\frac{\nu}{R^2}\overline{p}_{kn}t}}{J_1(\beta_n)\beta_n^3\frac{1-2De\overline{p}_{kn}+De^2\overline{p}_{kn}{}^2X}{(1-De\overline{p}_{kn})^2}}\right). \quad (21)$$

Analysis and results. Here, we first analyze the transition process in a Newtonian fluid in particular, and then analyze the transition processes in the motion of a mixture in general.

To perform Newtonian fluid analysis, the relaxation coefficient is set to zero. $De\to 0$, from the formula (21) we obtain:

$$\lim_{EL\to 0}v_x(r,t)=\lim_{De\to 0}\left(\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2\left(1-\frac{r^2}{R^2}-\right.\right.$$

$$\left.\left.-8\sum_{n=1}^{\infty}\sum_{k=1}^{2}\frac{J_0\left(\beta_n\frac{r}{R}\right)e^{-\frac{\nu}{R^2}\overline{p}_{kn}t}}{J_1(\beta_n)\beta_n^3\frac{1-2De\overline{p}_{kn}+De^2\overline{p}_{kn}{}^2X}{(1-De\overline{p}_{kn})^2}}\right)\right)= \quad (22)$$

$$=\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2\left(1-\frac{r^2}{R^2}-8\sum_{n=1}^{\infty}\frac{J_0\left(\beta_n\frac{r}{R}\right)e^{-\frac{\nu}{R^2}\beta_n^2t}}{J_1(\beta_n)\beta_n^3}\right),$$

$$\frac{v_x(0,t)}{v_x(0)} = \left(1 - 8\sum_{n=1}^{\infty} \frac{e^{-\frac{\nu}{R^2}\beta_n^2 t}}{J_1(\beta_n)\beta_n^3}\right).$$

It is not difficult to see that this solution is identical to the solution in [14,15]. As the relaxation coefficient approaches zero, the Maxwell model transitions to a Newtonian fluid. In this case, the solution obtained corresponds to the unsteady solution of the classical Poiseuille flow.

Conclusion. In this work, the problem of unsteady flow of a rheologically complex fluid obeying the Maxwell model under the influence of a constant pressure gradient in a cylindrical pipe was formulated mathematically and its analytical solution was obtained. Using Laplace–Carson transformations, the initial-boundary problem was solved, and the distribution of the velocity field in time and radial coordinates was expressed by Bessel functions. It is shown that the obtained general solution is reduced to the classical unsteady Poiseuille flow solution known for Newtonian fluids when the relaxation parameter tends to zero. This confirms the correctness and physical adequacy of the solution. It is also found that the elastic properties of the Maxwell model have a significant impact on the flow dynamics in unsteady processes. The analytical expressions obtained as a result serve as a reliable theoretical basis for a deep analysis of the processes of transition of rheologically complex fluids from unsteady to steady states, the influence of the Debord number and other dimensionless parameters in the subsequent stages. The analytical results of this problem will be considered in detail in a separate study in the next work.

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