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TWO-SOLITON SOLUTION OF THE FRACTIONAL MODIFIED KORTEWEG DE VRIES-SINE-GORDON EQUATION WITH A VARIABLE TIME-DEPENDENT COEFFICIENTS

Annotation. *In this paper, we investigate the fractional modified Korteweg de Vries-sine-Gordon (fmKdV-SG) equation with a variable time-dependent coefficients and show the inverse scattering transform method can also be used to obtain soliton solutions of this equation with a variable time-dependent coefficients. We have determined the time evolution of the scattering data for the Ablowitz-Kaup-Newell-Segur system associated with the solution of the fmKdV-SG equation with a variable time-dependent coefficients. Then, using the solution of the inverse scattering problem with respect to the time-dependent scattering data, we recover the desired solution of the fmKdV-SG equation with a variable time-dependent coefficients. For the one and two soliton case, explicit formulas for the solution of the problem under consideration are obtained. The solutions of the fmKdV-SG equation significance as a completely integrable system, paving the way for further exploration of its applications in mathematical physics and nonlinear wave phenomena. This research contributes valuable insights into nonlinear dynamics and soliton behavior, enhancing the understanding of multifaceted wave interactions in fractional equations.*

Keywords*: fractional modified Korteweg de Vries-sine-Gordon equation, variable time-dependent coefficients, dispersion relation, eigenvalue problem, inverse scattering transform method, soliton solutions, boundary conditions, integrable nonlinear evolution equations, Riesz fractional derivative.*

O'ZGARUVCHAN KOEFFITSIENTLI KASR TARTIBLI MODIFITSIRLANGAN KORTEVEG-DE FRIZ-SINUS-GORDON TENGLAMASINING IKKI SOLITONLI YECHIMI

Annotatsiya. *Ushbu maqolada o'zgaruvchan vaqtga bog'liq koeffitsiyentlarga ega bo'lgan kasr tartibli modifikatsiyalangan Korteveg-de Friz-sin-Gordon (fmKdV-SG) tenglamasi tadqiq qilinadi va ushbu tenglama uchun soliton yechimlarini olishda teskari masalalar usuli qo'llanilishi mumkinligi ko'rsatiladi. fmKdV-SG tenglamasining yechimi bilan bog'liq bo'lgan Ablovits-Kaup-Nyuell-Segur tizimi uchun tarqalish ma'lumotlarining vaqt bo'yicha evolutsiyasi aniqlanadi va vaqtga bog'liq tarqalish ma'lumotlariga asoslangan teskari masalalar usulidan foydalanib, fmKdV-SG tenglamasining istalgan yechimi qayta tiklanadi. Bir va ikki solitonli holatlar uchun muammoning aniq formulalari keltirilgan. fmKdV-SG tenglamasi to'liq integrallanuvchi tizim sifatida katta ahamiyatga ega bo'lib, matematik fizika va nochiziqli to'lqin hodisalaridagi qo'llanmalarning yanada chuqurroq o'rganilishi uchun yo'l ochadi. Ushbu tadqiqot kasr tartibli tenglamalardagi nochiziqli dinamika va soliton xatti-harakatlarini chuqurroq tushunishga hissa qo'shadi.*

Kalit so'zlar: *kasr tartibli modifikatsiyalangan Korteveg-de Friz-sin-Gordon tenglamasi, vaqtga bog'liq o'zgaruvchan koeffitsiyentlar, dispersiya munosabati, xos qiymat masalasi, teskari masalalar usuli, soliton yechimlari, chegaraviy shartlar, integrallanuvchi nochiziqli evolyutsion tenglamalar, Riesz kasr hosilasi.*

ДРОБНО-МОДИФИЦИРОВАННОЕ УРАВНЕНИЕ КОРТЕВЕГА-де-ФРИЗА -СИН-ГОРДОНА С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ

Аннотация. *В данной работе рассматривается дробно-модифицированное уравнение Кортевега-де Фриза-Син-Гордона(fmKdV-SG) с переменными, зависящими от времени, коэффициентами, и показано, что метод обратного рассеяния может быть применён для получения солитонных решений этого уравнения. Мы определяем временную эволюцию рассеяния для системы Абловица-Каупа-Ньюэл-ла-Сегора, связанной с решением уравнения Кортевега-де Фриза-Син-Гордона с переменными коэффициентами. Затем, используя решение обратной задачи рассеяния с учётом временных данных, мы восстанавливаем искомое решение данного уравнения. В случае одного и двух солитонов получены явные аналитические формулы решений. Уравнение Кортевега-де Фриза-Син-Гордона является полностью интегрируемой системой, что открывает новые перспективы для дальнейших исследований в области математической физики и нелинейных волновых явлений. Данное исследование вносит значительный вклад в понимание нелинейной динамики и поведения солитонов, улучшая представления о сложных волновых взаимодействиях в дробных уравнениях.*

Ключевые слова: *дробно-модифицированное уравнение Кортевега-де Фриза-Син-Гордона, переменные, зависящие от времени коэффициенты, дисперсионное соотношение, задача на собственные значения, метод обратного рассеяния, солитонные решения, граничные условия, интегрируемые нелинейные эволюционные уравнения, дробная производная Рисса.*

Introduction. The modified Korteweg de Vries-sine-Gordon(mKdV-SG) equation

$$u_{xt} + p_1\left(\frac{3}{2}\left(u_x\right)^2 u_{xx} + u_{xxxx}\right) - p_2 \sin u = 0, \tag{1}$$

is a non-linear partial differential equation which represents the well known modified Korteweg de Vries(mKdV) equation for $p_1 = 1$, $p_2 = 0$ and the sine-Gordon(SG) equation for $p_1 = 0$, $p_2 = 1$. It was introduced in [1, 2] as a model for describing nonlinear oscillations in an atomic lattice and shown that this equation is completely integrable and has multisoliton solutions. In [3], the mKdV-SG equation is used for describing the propagation of few-optical-cycle pulses in transparent media.

More recently, based on nonlinear extensions of Riesz fractional derivative Ablowitz et. al presented some new type of fractional nonlinear soliton equations, such as the fractional modiffied Korteweg-de Vries, fractional sine-Gordon, and fractional sinh-Gordon equations which are integrable in the sense of inverse scattering transform method (IST) [4]. For further development of the study of fractional nonlinear soliton equations which are integrable in the sense of IST, we refer to the works [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39].

In this paper, we investigate the fmKdV-SG equation with a variable time-dependent coefficients and show that the inverse scattering transform method can also be used to obtain soliton solutions of fmKdV-SG equation with a variable time-dependent coefficients. Namely, we show that the fmKdV-SG equation with a variable time-dependent coefficients is also an important theoretical model as it is a completely integrable system. We have determined the time evolution of the scattering data for the Ablowitz-Kaup-Newell-Segur(AKNS) system associated with the solution of the fmKdV-SG equation with a variable time-dependent coefficients. Then, using the solution of the inverse scattering problem with respect to the time-dependent scattering data, we recover the desired solution of the fmKdV-SG equation with a variable time-dependent coefficients.

Nonlinear fmKdV-SG equation with anomalous dispersions. In [5], it is used the scattering problem

$$v_x^{(1)} = -ikv^{(1)} + q(x,t)v^{(2)}, \tag{2}$$

$$v_x^{(2)} = ikv^{(2)} - q(x,t)v^{(1)}, \tag{3}$$

in order to define the mKdV-SG equation (1) which can also be defined by the equality

$$\sigma \mathbf{u}_t + 2A_0\left(\mathbf{L}^*\right)\mathbf{u} = 0 \tag{4}$$

with $\mathbf{u} = (-q(x,t),\ q(x,t))^T$, $q(x,t) = \frac{u_x(x,t)}{2}$ and $A_0\left(\mathbf{L}^*\right) = \frac{i}{4}\left(p_2\left(\mathbf{L}^*\right)^{-1} - 16p_1\left(\mathbf{L}^*\right)^3\right)$,

$$\mathbf{L}^* = \frac{1}{2i}\begin{pmatrix} \partial_x + 2q\left(I_q\right) & 2q\left(I_q\right) \\ -2q\left(I_q\right) & -\partial_x - 2q\left(I_q\right) \end{pmatrix},$$

where $I_- = \int_{-\infty}^{x} dy$ and $\mathbf{L}^*$ is the adjoint of the operator

$$\mathbf{L} = \frac{1}{2i}\begin{pmatrix} -\partial_x + 2q(I_+q) & -2q(I_+q) \\ 2q(I_+q) & \partial_x - 2q(I_+q) \end{pmatrix},$$

with $I_+ = \int_{x}^{\infty} dy$.

However, these definitions are inconvenient for the defining the nonlinear fractional modified Korteweg de Vries-sine-Gordon equation with a variable time-dependent coefficients. For this reason, we define the mKdV-SG equation with a variable time-dependent coefficients in terms of the scalar operator $L^A = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q^2 - q_x I_- q$ as follows

$$q_t + \Omega\left(L^A\right) q_x = 0, \tag{5}$$

with $q(x,t) = \frac{u_x(x,t)}{2}$. Where L^A is the adjoint of L:

$$L = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q^2 - qI_+ q_y,$$

and

$$\Omega\left(L^A\right) = \frac{p_2(t)}{4L^A} - 4p_1(t)L^A. \tag{6}$$

In order to define nonlinear fmKdV-SG equation with a variable time-dependent coefficients we use dispersion relation $w_f(k) = \left(\frac{p_2(t)}{k} - p_1(t)k^3\right)|k|^{2\varepsilon}$ of the linear fmKdV-SG equation with a variable time-dependent coefficients

$$u_{xt} + \left|-\partial_x^2\right|^{\varepsilon}\left(p_1(t)u_{xxxx} - p_2(t)u\right) = 0, \tag{7}$$

where $\left(-\partial_x^2\right)^{\epsilon}$, $\epsilon \in (0,1)$ is the Riesz fractional derivative.

Now, nonlinear fmKdV-SG equation with a variable time-dependent coefficients with Riesz fractional derivative can be obtained from (5) with $q(x,t) = \frac{u_x(x,t)}{2}$ and $\Omega_f\left(k^2\right) = \frac{w_f(2k)}{2k}$, namely, we will get the following nonlinear fmKdV-SG equation with a variable time-dependent coefficients

$$q_t + \Omega_f\left(L^A\right) q_x = 0, \tag{8}$$

where $\Omega_f\left(L^A\right) = \left(\frac{p_2(t)}{4L^A} - 4p_1(t)L^A\right)\left|4L^A\right|^{\epsilon}$.

The main aim of this work is to derive representations for the solution $u(x,t)$ of the problem (8) via inverse scattering method for the system (2) and (3).

The basic facts from scattering problem. Basic details on the scattering theory for the AKNS system are provided in [5,6].

We denote by $\phi(x;k,t)$, $\overline{\phi}(x;k,t)$ and $\psi(x;k,t)$, $\overline{\psi}(x;k,t)$ the solutions of the system

$$v_x^{(1)} = -ikv^{(1)} + q(x,t)v^{(2)}, \tag{9}$$

$$v_x^{(2)} = ikv^{(2)} - q(x,t)v^{(1)}, \tag{10}$$

satisfying the boundary conditions

$$\phi(x;k,t)e^{ikx} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\phi}(x;k,t)e^{-ikx} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x \to -\infty,$$

$$\psi(x;k,t)e^{-ikx} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{\psi}(x;k,t)e^{ikx} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x \to +\infty.$$

Then we have

$$\phi(x,k,t) = b(k,t)\psi(x,k,t) + a(k,t)\bar{\psi}(x,k,t), \tag{11}$$

$$\bar{\phi}(x,k,t) = \bar{a}(k,t)\psi(x,k,t) + \bar{b}(k,t)\bar{\psi}(x,k,t). \tag{12}$$

$$a(k,t) = W(\phi,\psi), \ \bar{a}(k,t) = W(\bar{\psi},\bar{\phi}), \quad b(k,t) = W(\bar{\psi},\phi), \quad \bar{b}(k,t) = W(\bar{\phi},\psi), \tag{13}$$

where $W(u,v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$.

Let

$$\tau(k,t) = \frac{1}{a(k,t)}, \quad \rho(k,t) = \frac{b(k,t)}{a(k,t)},$$

$$\bar{\tau}(k,t) = \frac{1}{\bar{a}(k,t)}, \quad \bar{\rho}(k,t) = \frac{\bar{b}(k,t)}{\bar{a}(k,t)},$$

$$\tilde{\rho}(k,t) = \frac{\bar{b}(k,t)}{a(k,t)}.$$

It is known that $\tau(k,t)$ is transmission coefficient, $\rho(k,t)$ is reflection coefficient and $\tilde{\rho}(k,t)$ is mixed reflection coefficient. When $r = -q$ the AKNS system (9) and (10) has eigenvalues $k_j = \xi_j + i\eta_j$, $\eta_j > 0$ and $\bar{k}_j = \bar{\xi}_j + i\bar{\eta}_j$, $\bar{\eta}_j < 0$, $j = 1,2,\ldots,J$ which zeros of a and $\bar{a}$ respectively, corresponding to bound states[6]. The following is the definition of the norming constants and the bound state eigenfunctions

$$\phi_j(x,t) = b_j(t)\psi_j(x,t), \quad \bar{\phi}_j(x,t) = \bar{b}_j(t)\bar{\psi}_j(x,t),$$

$$C_j(t) = \frac{b_j(t)}{a_j'(t)}, \bar{C}_j(t) = \frac{\bar{b}_j(t)}{\bar{a}_j'(t)}, \quad \tilde{C}_j(t) = \frac{\bar{b}_j(t)}{a_j'(t)},$$

where $b_j(t) = b\left(k_j,t\right)$ and $a_j'(t) = \partial_k a(k,t)\big|_{k=k_j}$.

The symmetry reductions for $r = -q$ are

$$\bar{\psi}(x;k,t) = \sigma_-\psi(x;-k,t), \quad \bar{\phi}(x;k,t) = \sigma_-^{-1}\phi(x;-k,t), \tag{14}$$

$$\bar{a}(k,t) = a(-k,t), \quad \bar{b}(k,t) = -b(-k,t), \tag{15}$$

where

$$\sigma_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_-^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let

$$\mathbf{M}(x,k,t) = e^{ikx}\phi(x,k,t), \quad \mathbf{N}(x,k,t) = e^{-ikx}\psi(x,k,t), \tag{16}$$

$\bar{\mathbf{M}}(x,k,t) = \sigma_-^{-1}\mathbf{M}(x,-k,t)$, $\bar{\mathbf{N}}(x,k,t) = \sigma_-\mathbf{N}(x,-k,t)$.

Then, by applying the Riemann-Hilbert method to inverse scattering problems, we discover

$$\mathbf{N}(x,k,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{j=1}^{J} \frac{C_j(t)e^{2ik_jx}}{k+k_j}\sigma_-^{-1}\mathbf{N}_j(x,t) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\xi,t)e^{2i\xi x}}{\xi+k}\sigma_-^{-1}\mathbf{N}(x,\xi,t)d\xi, \tag{17}$$

where $\mathbf{N}(x,k,t) = (N^{(1)}(x,k,t), N^{(2)}(x,k,t))^T$ and $\mathbf{N}_j(x,t) = \mathbf{N}\left(x,k_j,t\right)$. Evaluating this at k_ℓ for $\ell = 1,2,\ldots,J$, we obtain an equation for $\mathbf{N}(x,k_l,t) = \mathbf{N}_\ell(x,t)$

$$\mathbf{N}_\ell(x,t)=\begin{pmatrix}0\\1\end{pmatrix}-\sum_{j=1}^{J}\frac{C_j(t)e^{2ik_jx}}{k_\ell+k_j}\sigma_-^{-1}\mathbf{N}_j(x,t)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\xi,t)e^{2i\xi x}}{\xi+k_\ell}\sigma_-^{-1}\mathbf{N}(x,\xi,t)d\xi. \quad (18)$$

Similarly, we have

$$\mathbf{M}(x,k,t)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{j=1}^{J}\frac{\tilde{C}_j(t)e^{-2ik_jx}}{k+k_j}\sigma_-\mathbf{M}_j(x,t)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\rho}(\xi,t)e^{-2i\xi x}}{\xi+k}\sigma_-\mathbf{M}(x,\xi,t)d\xi, \quad (19)$$

$$\mathbf{M}_\ell(x,t)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{j=1}^{J}\frac{\tilde{C}_j(t)e^{-2ik_jx}}{k_\ell+k_j}\sigma_-\mathbf{M}_j(x,t)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\rho}(\xi,t)e^{-2i\xi x}}{\xi+k_\ell}\sigma_-\mathbf{M}(x,\xi,t)d\xi. \quad (20)$$

Now we can recover the solution at any time $q(x,t)$ from $\mathbf{N}(x,k,t)$ as

$$q(x,t)=2i\sum_{j=1}^{J}e^{2ik_jx}C_j(t)N_j^{(2)}(x,t)-\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(\xi,t)e^{2i\xi x}N^{(2)}(x,\xi,t)d\xi. \quad (21)$$

Time Evolution. We now consider the time evolution of the matrix eigenfunction $\mathbf{v}(x,t,k)$ in the form

$$\mathbf{v}_t=\begin{pmatrix}A & B\\ C & -A\end{pmatrix}\mathbf{v},$$

where A, B, C are functions of x, k, t which can not be explicitly presented in general for the fmKDV-SG equation with a variable time-dependent coefficients. However, their asymptotic properties $B(x,t;k),C(x,t;k)\to 0$ and $A(x,t,k)\to A_0(k)$ with $A_0(k)$ given as $x\to\pm\infty$ can be used to characterize the time evolution of the scattering data as

$$a(k,t)=a(k,0),\quad b(k,t)=b(k,0)e^{-2ik\left(\frac{p_2(t)}{4k^2}-4p_1(t)k^2\right)|2k|^{2\epsilon}t}, \quad (22)$$

$$\rho(k,t)=\rho(k,0)e^{-2ik\left(\frac{p_2(t)}{4k^2}-4p_1(t)k^2\right)|2k|^{2\epsilon}t}, \quad (23)$$

$$C_j(t)=C_j(0)e^{-2ik_j\Omega\left(k_j^2\right)t},\ j=1,2,\ldots,J, \quad (24)$$

where $\Omega\left(k^2\right)=\left(\frac{p_2(t)}{4k^2}-4p_1(t)k^2\right)|2k|^{2\epsilon}$.

Remark.

The obtained results completely define the time evolution of the spectral data, which allows us to solve the problem (8) by using the following algorithm:Let us given $q(x,0)$.

1. We find scattering data

$$\left\{\rho(k,0),k\in\mathbb{R},k_j(0),C_j(0),j=1,2,...,J\right\},$$

for (2) and (3) with the given with the given $q(x,0)$;

2. Using (22), (23) and (24), we obtain the time evolution of the scattering data

$$\left\{\rho(k,t),k\in\mathbb{R},k_j(t),C_j(t),j=1,2,...,J\right\}.$$

3. With the obtained scattering data, we uniquely define the function $N_j^{(2)}(x,t)$ by solving linear system of integral equations (17) and (18);

4. The function $q(x,t)$ can be obtain via the formula (21).

5. The solution $u(x,t)$ is defined by $q(x,t)=\frac{u_x(x,t)}{2}$.

The Two Soliton Solution. Let $\rho(k,t)=0$, $k\in\mathbb{R}$. For a given initial condition, the number of discrete eigenvalues $k_j=\xi_j+i\eta_j$, $j=1,2,\ldots,J$ gives the number of solitons. The general J-soliton solution can be reduced to solving a linear algebraic system. For simplicity we consider the two-soliton

solution, $J=2$. We find the two-soliton solution by constructing $\mathbf{N}(x,k,t)$ from equation (17) and (18) and then recovering $q(x,t)$ using (21).

Putting $\rho=0$, $J=2$ into equation (18) and taking $k_1=i\eta_1$, $k_2=i\eta$, $C_1(0)=-2i\eta_1 e^{2\eta_1 x_0}$ and $C_2(0)=-2i\eta_2 e^{2\eta_2 x_0}$ such that $C_1(t)=-2i\eta_1 e^{2\eta_1 x_0+2\eta_1\Omega\left(-\eta_1^2\right)t}$ and $C_2(t)=-2i\eta_2 e^{2\eta_2 x_0+2\eta_2\Omega\left(-\eta_2^2\right)t}$ gives an algebraic equation for $N_1(x,t)=\left(N_1^{(1)}(x,t),N_1^{(2)}(x,t)\right)^{\mathrm{T}}$ and $N_2(x,t)=\left(N_2^{(1)}(x,t),N_2^{(2)}(x,t)\right)^{\mathrm{T}}$

$$\begin{pmatrix}\mathbf{N}_1^{(1)}(k_1)\\ \mathbf{N}_1^{(2)}(k_1)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}-\frac{C_1(t)e^{2ik_1x}}{k_1+k_1}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\mathbf{N}_1^{(1)}(k_1)\\ \mathbf{N}_1^{(2)}(k_1)\end{pmatrix}-\frac{C_2(t)e^{2ik_2x}}{k_1+k_2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\mathbf{N}_2^{(1)}(k_2)\\ \mathbf{N}_2^{(2)}(k_2)\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{N}_2^{(1)}(k_2)\\ \mathbf{N}_2^{(2)}(k_2)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}-\frac{C_1(t)e^{2ik_1x}}{k_2+k_1}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\mathbf{N}_1^{(1)}(k_1)\\ \mathbf{N}_1^{(2)}(k_1)\end{pmatrix}-\frac{C_2(t)e^{2ik_2x}}{k_2+k_2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\mathbf{N}_2^{(1)}(k_2)\\ \mathbf{N}_2^{(2)}(k_2)\end{pmatrix}.$$

Solving these give

$$N_1^{(2)}(x,t)=\frac{1+\frac{5}{4}e^{z_1(x,t)+z_2(x,t)}+\frac{3}{2}e^{2z_1(x,t)}}{1+\frac{3}{2}e^{z_1(x,t)+z_2(x,t)}+\frac{1}{16}e^{2z_1(x,t)+2z_2(x,t)}+e^{2z_2(x,t)}+e^{2z_1(x,t)}},$$

$$N_2^{(2)}(x,t)=\frac{1+\frac{5}{4}e^{z_1(x,t)+z_2(x,t)}+\frac{3}{2}e^{2z_1(x,t)}}{1+\frac{3}{2}e^{z_1(x,t)+z_2(x,t)}+\frac{1}{16}e^{z_1(x,t)+2z_2(x,t)}+e^{2z_2(x,t)}+e^{z_1(x,t)}},$$

and then putting these components into equation (21) yields

$$q(x,t)=\frac{4\left(e^{z_1(x,t)}+3e^{z_2(x,t)}+\frac{25}{4}e^{2z_1(x,t)+z_2(x,t)}+\frac{35}{4}e^{z_1(x,t)+2z_2(x,t)}\right)}{1+\frac{3}{2}e^{z_1(x,t)+z_2(x,t)}+\frac{1}{16}e^{2z_1(x,t)+2z_2(x,t)}+e^{2z_2(x,t)}+e^{2z_1(x,t)}},$$

where $\eta_1=1$, $\eta_2=3$, $x_0=0$ $z_1(x,t)=2\eta_1\Omega\left(-\eta_1^2\right)t-2\eta_1 x$, $z_2(x,t)=2\eta_2\Omega\left(-\eta_2^2\right)t-2\eta_2 x$,

$\Omega\left(-\eta_1^2\right)=\left(4p_1(t)\eta_1^2-\frac{p_2(t)}{4\eta_1^2}\right)|\eta_1|^{2\varepsilon}$ and $\Omega\left(-\eta_2^2\right)=\left(4p_1(t)\eta_2^2-\frac{p_2(t)}{4\eta_2^2}\right)|\eta_2|^{2\varepsilon}$.

Conclusion. The inverse scattering transform method has been successfully used to obtain the soliton solutions of the fmKdV-SG equation with a variable time-dependent coefficients. We have shown that the fmKdV-SG equation with a variable time-dependent coefficients is also an important theoretical model as it is a completely integrable system via IST. We have determined the time evolution of the scattering data for the AKNS system associated with the solution of the fmKdV-SG equation with a variable time-dependent coefficients. Then, using the solution of the inverse scattering problem with respect to the time-dependent scattering data, we recover the desired solution of the fmKdV-SG equation with a variable time-dependent coefficients. For the two soliton case, explicit formulas for the solution of the problem under consideration are obtained.

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