Перспективы дальнейших исследований:

1. Случай переменного запаздывания $\tau(t)$;
2. Нелинейные функции источника $f(t,u)$;
3. Многомерные области по пространственным переменным;
4. Совместное восстановление коэффициентов уравнения и функции источника.

8. Заключение. В работе успешно поставлена и математически обоснована обратная задача по восстановлению функции источника в гиперболическом уравнении с постоянным запаздыванием. Выведено эквивалентное интегральное уравнение и строго доказана теоре-ма о существовании и единственности. Полученные результаты готовы к практическому применению при идентификации систем с задержкой.

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A FREE-BOUNDARY LOTKA–VOLTERRA COMPETITION SYSTEM WITH NONLINEAR ADVECTION AND TIME-DEPENDENT REMOVAL

Annotation. *We study a free-boundary model for two competing populations in a one-dimensional habitat, incorporating nonlinear self-advection and time-dependent proportional removal. The common boundary moves according to a weighted sum of the populations outward diffusive fluxes. Comparison estimates and a fixed-domain transformation yield positivity, uniform population bounds, monotone boundary expansion, an upper bound on boundary speed, and global existence and uniqueness of classical solutions. When removal sufficiently exceeds intrinsic growth, both populations decay exponentially and the boundary approaches a finite limit. The model accommodates seasonal or controlled removal.*

Keywords: *reaction–diffusion system, nonlinear advection, time-dependent removal, Lotka–Volterra competition, common free boundary, global existence, exponential extinction.*

CHIZIQSIZ ADVEKSIYA VA VAQTGA BOG'LIQ KAMAYTIRISHNI HISOBGA OLUVCHI ERKIN CHEGARALI LOTKA–VOLTERRA RAQOBAT SISTEMASI

Annotatsiya. *Bir o'lchamli hududda raqobatlashuvchi ikki populyatsiya uchun chiziqsiz adveksiya va vaqtga bog'liq proporsional kamaytirishni hisobga oluvchi erkin chegarali model o'rganiladi. Adveksiya har bir populyatsiyaning o'z zichligiga bog'liq bo'lib, umumiy chegaraning harakati ikkala populatsiyaning tashqariga yo'nalgan diffuzion oqimlari vaznli yig'indisi bilan belgilanadi. Taqqoslash usuli va masalani qo'zg'almas sohaga o'tkazish orqali zichliklarning musbatligi va tekis chegaralanganligi, hudud-*

ning monoton kengayishi, chegara tezligining chegaralanganligi hamda klassik yechimning global mavjudligi va yagonaligi isbotlanadi. Kamaytirish sur'atlari tabiiy o'sish sur'atlaridan yetarlicha yuqori bo'lganda, ikkala populatsiya zichligi eksponensial ravishda nolga, chegara esa chekli limitga intiladi. Model mavsumiy yoki boshqariladigan kamaytirishni hisobga olish imkonini beradi.

Kalit so'zlar: *reaksiya–diffuziya tizim, chiziqsiz adveksiya, vaqtga bog'liq kamaytirish, Lotka–Volterra raqobati, umumiy erkin chegara, global mavjudlik, eksponensial so'nish.*

СИСТЕМА КОНКУРЕНЦИИ ЛОТКИ–ВОЛЬТЕРРЫ СО СВОБОДНОЙ ГРАНИЦЕЙ, НЕЛИНЕЙНОЙ АДВЕКЦИЕЙ И ЗАВИСЯЩИМ ОТ ВРЕМЕНИ ИЗЪЯТИЕМ

Аннотация. *Рассматривается модель конкуренции двух популяций в одномерной области со свободной границей, нелинейной самоадвекцией и зависящим от времени пропорциональным изъятием. Движение общей границы определяется взвешенной суммой диффузионных потоков обеих популяций, направленных наружу. Метод сравнения и преобразование к фиксированной области позволяют доказать положительность и равномерную ограниченность плотностей популяций, монотонное расширение области и ограниченность скорости границы, а также глобальное существование и единственность классического решения. При достаточно сильном преобладании изъятия над естественным ростом обе популяции экспоненциально вымирают, а граница стремится к конечному пределу. Модель допускает сезонное или регулируемое изъятие.*

Ключевые слова: *система реакции–диффузии, нелинейная адвекция, зависящее от времени изъятие, конкуренция Лотки–Вольтерры, общая свободная граница, глобальное существование, экспоненциальное вымирание.*

1. **Introduction.** Reaction-diffusion equations are a standard tool for describing the spatial interaction of biological populations. The classical Lotka-Volterra framework combines intrinsic growth, self-limitation and interspecific competition, while diffusion represents random dispersal. Mathematical foundations and ecological interpretations of such models are discussed in [1, 2]. When the occupied habitat itself changes in time, a free boundary gives a natural description of the propagation front. The logistic free boundary model and its spreading-vanishing dichotomy were developed in [3], and related biological free boundary mechanisms were surveyed in [4].

For interacting populations, free boundary formulations have been studied for predator-prey and competition systems. Lin [5] considered a predator-prey model; Guo and Wu [6] and Wang and Zhao [7] analyzed two-species competition systems. Advection and nonlinear transport may substantially change the movement of a population front. Relevant results for advection-diffusion free boundary equations and systems can be found in [8-11]. A quasilinear competition system with nonlinear convection and a common free boundary was investigated in [10], where a priori Holder estimates, uniqueness and existence were established.

2. Formulation of the new problem

Let $u(t,x)$ and $v(t,x)$ denote the densities of two competing populations at time t and position x. Their common habitat is the interval $0 < x < h(t)$, where $h(t)$ is an unknown right boundary. For a fixed $T > 0$, define

$$Q_T = \{(t,x): 0 < t \le T,\ 0 < x < h(t)\}. \tag{1}$$

We consider the following nonautonomous quasilinear system:

$$u_t - d_1 u_{xx} - m_1 u u_x = u\big(a_1(t) - q_1(t) - b_1 u - c_1 v\big), \qquad (t,x) \in Q_T. \tag{2}$$

$$v_t - d_2 v_{xx} - m_2 v v_x = v\big(a_2(t) - q_2(t) - b_2 v - c_2 u\big), \qquad (t,x) \in Q_T. \tag{3}$$

The initial conditions are

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x), \qquad 0 \le x \le h_0. \tag{4}$$

At the fixed left endpoint we impose zero flux, while both densities vanish at the moving front:

$$u_x(t,0) = v_x(t,0) = 0, \qquad 0 < t \le T. \tag{5}$$

$$u(t,h(t)) = v(t,h(t)) = 0, \qquad 0 < t \le T. \tag{6}$$

The common front obeys the Stefan condition

$$h'(t) = -\mu\left[d_1 u_x(t,h(t)) + \rho d_2 v_x(t,h(t))\right], \qquad h(0) = h_0. \tag{7}$$

Here $d_i > 0$ are diffusion coefficients, $m_i \ge 0$ measure nonlinear self-advection, $b_i > 0$ are intraspecific competition coefficients, $c_i > 0$ are interspecific competition coefficients, and $\mu, \rho > 0$ describe the response of the front to population flux. The functions $a_i(t)$ are intrinsic growth rates and $q_i(t)$ are proportional removal rates. Thus $r_i(t) = a_i(t) - q_i(t)$ is the effective per-capita growth rate at low density.

We use the following hypotheses.

I. The functions a_i, q_i belong to $C^{\alpha/2}([0,\infty))$ for some $0 < \alpha < 1$, and there are constants $a_i^+, q_i^+ > 0$ such that $0 < a_i(t) \le a_i^+$ and $0 \le q_i(t) \le q_i^+$.

II. The initial functions u_0, v_0 belong to $C^{2+\alpha}([0,h_0])$, are positive in $(0,h_0)$, and satisfy $u_0(0) = v_0(0) = 0$, $u_0(h_0) = v_0(h_0) = 0$, $u_0(h_0) < 0$, and $v_0(h_0) < 0$.

III. All constant model parameters appearing in (2)-(7) are positive, except that m_i may be zero.

3. A priori estimates

The first theorem supplies bounds that are independent of the local existence time. These estimates are the key to continuation of a classical solution.

Theorem 1. Suppose I-III hold and (u,v,h) is a classical solution of (1)-(7). Then

$$0 < u(t,x) \le M_1, \qquad 0 < v(t,x) \le M_2, \qquad (t,x) \in Q_T, \tag{8}$$

where

$$M_1 = \max\left\{ \|u_0\|_\infty, \frac{a_1^+}{b_1} \right\}, \qquad M_2 = \max\left\{ \|v_0\|_\infty, \frac{a_2^+}{b_2} \right\}. \tag{9}$$

Moreover, $h'(t) > 0$ and there is a constant $C_h > 0$, depending only on the data and the bounds in (9), such that

$$0 < h'(t) \le C_h, \qquad 0 < t \le T.(1) \tag{10}$$

Proof. Since the reaction terms vanish when the corresponding density is zero, the strong maximum principle and the initial positivity imply $u > 0$ and $v > 0$ in Q_T. At an interior positive maximum of u, the terms $-c_1 uv$ and $-q_1(t)u$ are nonpositive. Comparison with the scalar logistic equation $z' = z(a_1^+ - b_1 z)$ yields the first bound in (8). The second follows in the same way. These arguments are standard for competitive parabolic systems; see [1, 2].

The Hopf boundary lemma applied at $x = h(t)$ gives $u_x(t,h(t)) < 0$ and $v_x(t,h(t)) < 0$. Equation (7) therefore implies $h'(t) > 0$. To obtain the upper estimate, one may introduce linear barriers in a boundary strip, or equivalently use the boundary-gradient estimate for uniformly parabolic equations after straightening the free boundary. The coefficient bounds and (8) give

$$-u_x(t,h(t)) \le K_1, \qquad -v_x(t,h(t)) \le K_2, \tag{11}$$

where K_1 and K_2 depend only on the prescribed data on $[0,T]$. Substitution into (7) yields $C_h = \mu(d_1K_1 + \rho d_2K_2)$, proving (10).

Corollary 1. For every finite $T > 0$, the moving boundary satisfies

$$h_0 < h(t) \le h_0 + C_h t, \qquad 0 < t \le T. \tag{12}$$

This follows immediately by integration of (10).

4. Transformation to a fixed domain and local solvability

Set $y = x / h(t)$ and define $U(t,y) = u(t,h(t)y)$ and $V(t,y) = v(t,h(t)y)$. The moving cylinder Q_T is mapped onto $\Omega_T = (0,T] \times (0,1)$. Direct differentiation gives

$$u_t = U_t - \frac{yh'(t)}{h(t)}U_y, \qquad u_x = \frac{1}{h(t)}U_y, \qquad u_{xx} = \frac{1}{h^2(t)}U_{yy}. \tag{13}$$

The corresponding identities hold for v. Hence the transformed system is

$$U_t - \frac{d_1}{h^2(t)}U_{yy} - \frac{yh'(t) + m_1U}{h(t)}U_y = U\left(a_1(t) - q_1(t) - b_1U - c_1V\right). \tag{14}$$

$$V_t - \frac{d_2}{h^2(t)}V_{yy} - \frac{yh'(t) + m_2V}{h(t)}V_y = V\left(a_2(t) - q_2(t) - b_2V - c_2U\right). \tag{15}$$

The fixed-boundary conditions are

$$U_y(t,0) = V_y(t,0) = 0, \qquad U(t,1) = V(t,1) = 0. \tag{16}$$

The Stefan condition becomes

$$h'(t) = -\frac{\mu}{h(t)}\left[d_1U_y(t,1) + \rho d_2V_y(t,1)\right]. \tag{17}$$

Because $h(t) \ge h_0$, the principal coefficients $d_i / h^2(t)$ remain strictly positive on every finite interval. The transformed problem is therefore uniformly parabolic as long as h, U and V remain in bounded Holder sets.

Theorem 2. Under H1-H3, there exists $T_0 > 0$ such that problem (1)-(7) has a unique classical solution satisfying

$$u, v \in C^{1+\alpha/2,2+\alpha}(Q_{T_0}), \qquad h \in C^{1+\alpha/2}([0,T_0]). \tag{18}$$

Proof. On the fixed cylinder, freeze h, U and V in the lower-order coefficients of (14)-(17). Linear parabolic theory in Holder spaces [10, 11] gives a solution operator. For sufficiently small T_0, the operator maps a closed ball determined by the initial norms into itself and is compact and continuous. The Leray-Schauder principle [10] gives a fixed point. Alternatively, the contraction argument used for free boundary reaction-diffusion systems in [3, 5] applies after reducing T_0. To prove uniqueness, subtract two solutions on the fixed cylinder, estimate the differences by parabolic Schauder estimates, and use (17) to control the difference of the two boundaries. A small-time contraction follows, and continuation gives uniqueness throughout the common interval of existence.

5. Global continuation

Theorem 3. The local solution from Theorem 2 extends uniquely to every finite time $T > 0$.

Proof. Let $T_{\max}$ be the maximal existence time and suppose $T_{\max} < \infty$. The density estimates (8) prevent blow-up of u and v. Estimates (10)-(12) show that h and h' remain bounded and that $h(t) \geq h_0 > 0$. Thus the transformed equations (14)-(15) remain uniformly parabolic. Interior and boundary Schauder estimates [10, 11], followed by the Stefan relation (17), yield for each $0 < \tau < T_{\max}$ a bound

$$\|U\|_{C^{1+\alpha/2,2+\alpha}([\tau,T_{\max})\times[0,1])} + \|V\|_{C^{1+\alpha/2,2+\alpha}([\tau,T_{\max})\times[0,1])} + \|h\|_{C^{1+\alpha/2}([\tau,T_{\max}])} \leq C_\tau. \tag{19}$$

Consequently, the solution data converge in the required compatibility spaces as t approaches $T_{\max}$. The local existence argument can be restarted at $T_{\max}$, contradicting maximality. Hence $T_{\max} = \infty$.

6. Exponential extinction under dominant removal

We now assume that removal uniformly exceeds intrinsic growth. Define

$$\delta_i = \inf_{t\geq 0}\big(q_i(t) - a_i(t)\big), \qquad i = 1, 2. \tag{20}$$

Theorem 4. If $\delta_1 > 0$ and $\delta_2 > 0$, then every global classical solution satisfies

$$\|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty e^{-\delta_1 t}, \qquad \|v(t,\cdot)\|_\infty \leq \|v_0\|_\infty e^{-\delta_2 t}. \tag{21}$$

Furthermore, $h(t)$ converges to a finite limit h_∞.

Conclusion. A nonautonomous free boundary model has been formulated for two competing populations with nonlinear self-advection and time-dependent proportional removal. The use of bounded time-dependent coefficients preserves the comparison structure of the autonomous system. Uniform density estimates and boundary-gradient bounds provide local and global well-posedness. If removal exceeds the largest possible intrinsic growth rate for each species, both populations decay exponentially and the common habitat approaches a finite limiting length. The explicit estimates (13)-(17) connect management intensity with the remaining expansion of the habitat and the time required to reach a prescribed density level. Future work may address sign-changing net growth, periodic persistence thresholds and species-dependent free boundaries.

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НАПРЯЖЁННО-ДЕФОРМИРОВАННОЕ СОСТОЯНИЕ И РАССЕЯНИЕ ВОЛН В ВЯЗКОУПРУГОМ ПОЛУПРОСТРАНСТВЕ С ЦИЛИНДРИЧЕСКИМ ТОННЕЛЕМ БЕЗ ОБДЕЛКИ ПОД ДЕЙСТВИЕМ ДВИЖУЩЕЙСЯ НАГРУЗКИ

Аннотация. *Исследуется задача о напряжённо-деформированном состоянии линейно-вязкоупругого однородного изотропного полупространства с цилиндрической полостью (тоннелем без обделки), ось которой расположена на некоторой глубине параллельно свободной поверхности. К контуру полости приложена нормальная или касательная нагрузка, равномерно движущаяся с постоянной дозвуковой скоростью, меньшей скорости волны Рэлея. В рамках пространственной краевой задачи динамической теории вязкоупругости построено точное аналитическое решение, основанное на разложении волновых потенциалов по плоским волнам, принципе суперпозиции волновых полей и их переразложении в ряды по цилиндрическим функциям. Данный подход позволяет точно учитывать процессы многократного отражения волн от дневной поверхности и их рассеяния на контуре полости без пространственной дискретизации среды. Выявлены закономерности распределения динамических напряжений и перемещений в зависимости от скорости движения нагрузки, глубины заложения полости и вязкоупругих параметров среды. Полученные результаты могут служить эталоном для оценки динамической прочности малых транспортных тоннелей.*

Ключевые слова: *вязкоупругое полупространство, тоннель без обделки, движущаяся нагрузка, дифракция волн, напряжённо-деформированное состояние, аналитическое решение, цилиндрический волновой потенциал.*

HARAKATLANUVCHI YUK TA'SIRIDA QOPLAMASIZ SILINDRIK TUNNELGA EGA QOVUSHQOQ-ELASTIK YARIM FAZONING KUCHLANGANLIK-DEFORMATSIYA HOLATI VA UNDA TO'LQINLARNING TARQALISHI

Annotatsiya. *O'qi erkin sirtga parallel ravishda muayyan chuqurlikda joylashgan silindrik bo'shliqqa (qoplamasiz tunnelga) ega chiziqli qovushqoq-elastik, bir jinsli va izotrop yarim fazoning kuchlanish-deformatsiya holati masalasi tadqiq etiladi. Bo'shliq konturiga Reley to'lqini tezligidan kichik bo'lgan doimiy tovushosti tezlik bilan tekis harakatlanuvchi normal yoki urinma yuk qo'yilgan. Qovushqoq-elastiklik dinamik nazariyasining fazoviy chegaraviy masalasi doirasida to'lqin potensiallarini yassi to'lqinlar bo'yicha yoyish, to'lqin maydonlarining superpozitsiya prinsipi va ularni silindrik funksiyalar bo'yicha qatorlarga qayta yoyishga asoslangan aniq analitik yechim qurilgan. Ushbu yondashuv muhitni fazoviy diskretlashtirmasdan, to'lqinlarning yer sirtidan ko'p marta qaytishi va bo'shliq konturida sochilishi jarayonlarini aniq hisobga olish imkonini beradi. Dinamik kuchlanishlar va ko'chishlarning yuk harakati tezligiga, bo'shliqning joylashish chuqurligiga hamda muhitning qovushqoq-elastik parametrlariga bog'liq holda taqsimlanish qonuniyatlari aniqlangan. Olingan natijalar kichik transport tunnellarining dinamik mustahkamligini baholash uchun etalon bo'lib xizmat qilishi mumkin.*

Kalit so'zlar: *qovushqoq-elastik yarim fazo, qoplamasiz tunnel, harakatlanuvchi yuk, to'lqinlar difraksiyasi, kuchlanish-deformatsiya holati, analitik yechim, silindrik to'lqin potensiali.*