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$\mathbb{C}\setminus\mathbb{N}$ - PARABOLIK KO‘PXILLIKDA POLINOMLAR YOYILMASI

Annotatsiya. *Ushbu maqolada bir o‘lchamli Shteyn ko‘pxilliklarida, ya’ni ochiq Riman sirtda aniqlangan golomorf funksiyalarning maxsus sinfi d – darajali ρ – polinomlar o‘rganilgan bo‘lib, unda qamrov funksiyasining logarifmik singulyarliklari orqali funksiyalarning qutb tartiblari bilan bog‘liqligi tahlil qilingan. $\mathbb{C}\setminus\mathbb{N}$ ko‘pxillikda Loran qatori yordamida golomorf funksiyalarning lokal xossalari o‘rganilib, Liuvill teoremasi asosida har qanday d – darajali ρ – polinomni chekli ratsional funksiyalar yig‘indisi ko‘rinishida ifodalash mumkinligi isbotlangan. Shuningdek, kompleks tekislikdan ayrim nuqtalar chiqarib tashlangan holda hosil qilingan bir o‘lchamli ko‘pxillik misolida olingan natijalar asoslab berilgan.*

Kalit so‘zlar: *Shteyn ko‘pxilligi, golomorf funksiya, ρ – polinom, Loran qatori, qutb nuqta, Liuvill teoremasi, ratsional funksiya.*

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РАЗЛОЖЕНИЕ ПОЛИНОМОВ НА $\mathbb{C}\setminus\mathbb{N}$ - ПАРАБОЛИЧЕСКОМ МНОГООБРАЗИИ

Аннотация. *В данной работе изучается специальный класс голоморфных функций– ρ -полиномы степени d, определенные на одномерных многообразиях Штейна, а именно на открытых римановых поверхностях, в которых анализируется связь порядков полюсов функций через логарифмические особенности покрывающей функции. С помощью ряда Лорана изучены локальные свойства голоморфных функций на многообразии $\mathbb{C}\setminus\mathbb{N}$ и на основе теоремы Лиувилля доказано, что любой ρ -полином степени d можно представить в виде суммы конечных рациональных функ-*

ций. Также полученные результаты обоснованы на примере одномерного многообразия, образованного путем удаления некоторых точек из комплексной плоскости.

Ключевые слова: *многообразие Штейна, голоморфная функция, ρ-полином, ряд Лорана, полюс, теорема Лиувилля, рациональная функция.*

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EXPANSION OF POLYNOMIALS ON THE $\mathbb{C}\setminus\mathbb{N}$ - PARABOLIC MANIFOLD

Annotation. *In this paper, a special class of holomorphic functions - ρ-polynomials of degree d defined on one-dimensional Stein manifolds, namely, open Riemann surfaces—is studied. The relationship between the pole orders of the functions is analyzed through the logarithmic singularities of the covering function. The local properties of holomorphic functions on the $\mathbb{C}\setminus\mathbb{N}$ manifold are investigated using the Laurent series. Based on Liouville's theorem, it is proved that any ρ-polynomial of degree d can be expressed as a sum of finite rational functions. Furthermore, the obtained results are justified by taking a one-dimensional manifold formed by removing certain points from the complex plane as an example.*

Keywords: *Stein manifold, holomorphic function, ρ-polynomial, Laurent series, pole, Liouville's theorem, rational function.*

Kompleks analiz va ko‘pxilliklar nazariyasida Shteyn ko‘pxilliklari muhim ahamiyatga ega. Ayniqsa, parabolik Shteyn ko‘pxilliklarda aniqlangan golomorf funksiyalarni o‘rganish, ularning o‘sish tartibi va strukturaviy xossalarini aniqlash zamonaviy matematikada dolzarb masalalardan biridir. Bunday fazolarda polinom tushunchasi klassik ko‘phadlardan farqli ravishda maxsus baholash shartlari orqali aniqlanadi. Ushbu maqolada parabolik Shteyn ko‘pxilliklarida aniqlangan golomorf funksiyalarning maxsus sinfi - $\rho-$polinomlar qaraladi. Qamrov funksiyasining logarifmik singulyarliklari orqali funksiyaning qutb tartiblari orasidagi bog‘liqlik tahlil qilingan. Shuningdek, Loran qatori va Liuvill teoremasi yordamida bunday polinomlarning chekli ratsional yig‘indi ko‘rinishida ifodalanishi masalasi o‘rganilgan.

Agar n o‘lchamli X Shteyn ko‘pxilligida aniqlangan chegaralangan plyurisubgarmonik funksiyalar faqat o‘zgarmas funksiyalardan iborat bo‘lsa, u holda bu ko‘pxillikka parabolik deyiladi.

Bu X fazoning paraboliklig quyidagiga ekvivalent. Agar

$$\rho(z)\in psh(X)$$

bo‘lib, $\exists C\in\mathbb{R}$ soni uchun $\rho(z)<C$ bo‘lsa, u holda X da bo‘ladi.

1-ta’rif. Agar n o‘lchamli X Shteyn ko‘pxilligida shunday $\rho(z)$ plyurisubgarmonik funksiya topilib, quyidagi shartlar bajarilsa,

1. $\forall c\in\mathbb{R},\quad D_c=\left\{z\in X:\rho(z)<c\right\}\subset\subset X;$

2. $\rho(z)$ funksiya biror $K\subset X$ kompaktdan tashqarida maksimal bo‘lsin. Ya’ni, $X\setminus K$ da $(dd^c\rho)^n=0$ bo‘lsa, u holda X ko‘pxillikka $S-$*parabolik* deyiladi (3).

Agar yuqoridagi ta’rifdagi shartlarga qo‘shimcha ravishda $\rho\in C(X)$ uzluksiz funksiya bo‘lsa, u holda X ko‘pxillik S^*-*parabolik* deyiladi [4], [7].

2-ta’rif. Agar $(X,\rho)-$ Shteyn ko‘pxilligi S^* – parabolik ko‘pxillik bo‘lib, $\forall f\in\mathcal{O}(X)$ golomorf funksiya uchun shunday d va c musbat sonlar topilib,

$$\ln\left|f(z)\right|\le d\rho^+(z)+c\ ,\ \forall z\in X \qquad (1)$$

baholashni qanoatlantirsa, $f(z)$ funksiyaga $\rho-$ *polinom* deyiladi. Bu yerda c faqat $f(z)$ funksiyaga bog‘liq o‘zgarmas.

(1) tengsizlikni qanoatlantiruvchi d ning eng kichik butun qiymatiga f polinomning darajasi deyiladi va d natural son uchun X dagi barcha darajasi d dan oshmaydigan $\rho-$polinomlar to‘plami $\mathcal{P}_\rho^d(X)$ orqali belgilaymiz.

Parabolik ko‘pxilliklar ko‘pgina olimlar tomonidan o‘rganilgan. Bir o‘lchamli kompleks ko‘pxilliklar, ya’ni Riman sirtlari klassifikatsiyasi L.Sario va M.Nakai [5] monografiyasida batafsil yoritiladi. Xususan, parabolik, S–parabolik, S^*–parabolik ko‘pxilliklar uchun keltirilgan ta’riflar ekvivalent bo‘lishi ko‘rsatilgan. W.Stoll [6] ishlarida parabolik ko‘pxilliklarda Nevanlinna nazariyasi asoslari ishlab chiqilgan. [1], [2] ishlarda parabolik ko‘pxilliklarga aniq misollar va ularning xususiyatlari o‘rganilgan.

Bizga kompleks tekislikdan natural sonlarga mos nuqtalarni chiqarib tashlash orqali hosil qilingan ushbu $X = \mathbb{C} \setminus \mathbb{N}$ bir o‘lchamli kompleks ko‘pxillik berilgan bo‘lsin. $X = \mathbb{C} \setminus \mathbb{N}$ ko‘pxillikda aniqlangan ushbu

$$\rho(z) = 2\ln|z| - \sum_{n=1}^{\infty} \frac{1}{2^n} \ln|z-n|$$

qamrov funksiyani qaraymiz. Bu funksiya X da aniqlangan va subgarmonik bo‘lib, $X \setminus \{0\}$ to‘plamda garmonik funksiya bo‘ladi. Ya’ni (X, ρ) ko‘pxillik *–parabolik.*

Ushbu maqolaning asosiy natijasi quyidagi tasdiqdan iborat.

1-teorema. (X, ρ) *ko‘pxillikda har qanday d- darajali* ρ *—polinom quyidagicha*

$$f(z) = \sum_{j_0+2j_1+4j_2+\ldots+2^n j_n=0}^{d} C_{j_0 j_1 \ldots j_n} z^{j_0} \cdot \frac{1}{(z-1)^{j_1}} \cdot \ldots \cdot \frac{1}{(z-m)^{j_n}}$$

chekli yoyilmaga ega bo‘ladi. Bu yerda $m = \left[\log_2 d\right]$.

Isbot. $f(z)$ funksiya $X = \mathbb{C} \setminus \mathbb{N}$ da golomorf bo‘lgani uchun har bir $z = k$, $k = 1, 2, \ldots, m$ nuq-talar atrofida Loran qatoriga yoyiladi. Dastlab, bu funksiyani $z = 1$ nuqta atrofida

$$f(z) = \sum_{k=-\infty}^{+\infty} c_k (z-1)^k \tag{2}$$

Loran qatoriga yoyamiz. $f(z)$ - d - darajali ρ —polinom bo‘lgani uchun 2-ta’rifga ko‘ra, shunday $C_1 > 0$ son topilib, ushbu

$$\ln|f(z)| \le C_1 + d\rho(z) = C_1 + d\left(2\ln|z| - \sum_{n=1}^{\infty} \frac{1}{2^n} \ln|z-n|\right)$$

munosabat o‘rinli bo‘ladi. Bundan esa $z = 1$ nuqtaning yetarlicha kichik atrofida

$$\ln|f(z)| \le C_1' - \frac{d}{2}\ln|z-1|, \text{ ya’ni}$$

$$\ln\left(|f(z)| \cdot |z-1|^{\frac{d}{2}}\right) \le C_1'$$

tengsizlik bajarilishini ko‘rish mumkin. Bu esa $f(z)$ funksiya $z = 1$ da $\left[\frac{d}{2}\right]$ tartibli qutbga ega bo‘lishini bildiradi. (2) dan ushbu

$$g_1(z) = \frac{c_{-1}^{(1)}}{z-1} + \frac{c_{-2}^{(1)}}{(z-1)^2} + \ldots + \frac{c_{-\left[\frac{d}{2}\right]}^{(1)}}{(z-1)^{\left[\frac{d}{2}\right]}}$$

ratsional funksiyani ayrib, $f_1(z) = f(z) - g_1(z)$ funksiyani hosil qilamiz. Bu funksiya $z = 1$ nuqta atrofida golomorf bo‘ladi.

Endi $f_1(z)$ funksiyani $z = 2$ atrofida quyidagi

$$f_1(z)=\sum_{k=-\infty}^{+\infty} c_k (z-2)^k \tag{3}$$

Loran qatoriga yoyib, $f_1(z)$ funksiya uchun ρ – polinom ta'rifiga ko'ra, shunday $C_2>0$ son mavjud bo'lib,

$$\ln|f_1(z)|\le C_2+d\rho(z)=C_2+d\left(2\ln|z|-\sum_{n=1}^{\infty}\frac{1}{2^n}\ln|z-n|\right)$$

tengsizlik o'rinli bo'lishi hamda $z=2$ nuqtaning yetarlicha kichik atrofida ushbu

$$\ln|f_1(z)|\le C_2'-\frac{d}{2^2}\ln|z-2|, \text{ ya'ni}$$

$$\ln\left(|f_1(z)|\cdot|z-2|^{\frac{d}{2^2}}\right)\le C_2'$$

tengsizlik o'rinli bo'lishini bilamiz. Bu esa $f_1(z)$ funksiya $z=2$ da $\left[\frac{d}{2^2}\right]$ tartibli qutbga ega bo'lishini bildiradi. (3) dan ushbu

$$g_2(z)=\frac{c_{-1}^{(2)}}{z-2}+\frac{c_{-2}^{(2)}}{(z-2)^2}+\ldots+\frac{c_{-\left[\frac{d}{2^2}\right]}^{(2)}}{(z-2)^{\left[\frac{d}{2^2}\right]}}$$

ratsional funksiyani ayrib, $f_2(z)=f_1(z)-g_2(z)=f(z)-g_1(z)-g_2(z)$ funksiyani hosil qilamiz. Bu funksiya $z=2$ nuqta atrofida golomorf bo'ladi. Bu jarayonni xuddi shu tarzda davom qildirib, chekli qadamda to'xtatamiz. Agar bu jarayonni $z=m$ nuqtagacha davom qildirsak, bu nuqta atrofida golomorf bo'lgan ushbu

$f_m(z)=f_{m-1}(z)-g_m(z)=f(z)-g_1(z)-g_2(z)-\ldots-g_m(z)$, bu yerda

$$g_m(z)=\frac{c_{-1}^{(m)}}{z-m}+\frac{c_{-2}^{(m)}}{(z-m)^2}+\ldots+\frac{c_{-\left[\frac{d}{2^m}\right]}^{(m)}}{(z-m)^{\left[\frac{d}{2^m}\right]}}$$

ratsional funksiyani hosil qilamiz. Natijada $f_m(z)$ funksiya uchun ham ρ – polinom ta'rifini qo'llasak, shunday $C_{m+1}>0$ son uchun ushbu

$$\ln|f_m(z)|\le C_{m+1}+d\rho(z)=C_{m+1}+d\left(2\ln|z|-\sum_{n=1}^{\infty}\frac{1}{2^n}\ln|z-n|\right)$$

tengsizlik o'rinli bo'lishidan $z=m$ nuqtaning yetarlicha kichik atrofida ushbu

$$\ln|f_m(z)|\le C_{m+1}'-\frac{d}{2^m}\ln|z-m|, \text{ ya'ni}$$

$$\ln\left(|f_m(z)|\cdot|z-m|^{\frac{d}{2^m}}\right)\le C_{m+1}'$$

tengsizlik o'rinli bo'ladi.

Agar $\frac{d}{2^m}<1$ bo'lsa, $f_m(z)$ funksiya uchun $z=m$ nuqta chetlatish mumkin bo'lgan maxsus nuqta bo'ladi, ya'ni, $f_m(z)$ning Loran qator yoyilmasida bosh qismi bo'lmaydi va $f(z)$ funksiya $m+1, m+2, \ldots$ nuqtalarda golomorf davom etadi ([8], 5-bob, 14-§, 1-teorema). Natijada

$$f_m(z)=f(z)-g_1(z)-g_2(z)-\ldots-g_m(z) \tag{4}$$

funksiya $\mathbb{C}$ ga golomorf davom etadi, ya'ni, $|z|>R$, $R>m$ sohada butun funksiya bo'ladi. Natijada

$$\ln|f_m(z)|\le C'+d\rho(z)=C'+2d\ln|z|$$

bo'lishidan $f_m(z)=c_0+c_1z+\ldots+c_dz^d$ ko'rinishdagi d-darajali ko'phad bo'ladi. $f(z)$ funksiya $\mathbb{C}\setminus\mathbb{N}$ da golomorf bo'lib, cheksizlikda $|z|^d$ dan oshmaydi. Liuvill ([8], 2-bob, 6-§, 2-teorema) teoremasiga ko'ra, $f_m(z)$ darajasi d dan katta bo'lmagan ko'phad bo'ladi. (4) dan

$$f(z)=f_m(z)+g_1(z)+g_2(z)+\ldots+g_m(z)$$

tenglik hosil bo'ladi. Bu tenglikning o'ng qismini $g_i(z)$, $i=1,2,\ldots,m$ larni umumiy maxrajga keltirib, soddalashtirsak ushbu

$$f(z)=\sum_{j_0+2j_1+4j_2+\ldots+2^nj_n=0}^{d}C_{j_0j_1\ldots j_n}z^{j_0}\cdot\frac{1}{(z-1)^{j_1}}\cdot\ldots\cdot\frac{1}{(z-m)^{j_n}}$$

tenglik hosil bo'ladi. Teorema isbot bo'ldi.

[9] ishda $\mathbb{C}\setminus\mathbb{N}$ – parabolik Riman sirtidagi polinomlar fazosi o'lchami, ya'ni, $\dim\mathscr{P}_\rho^d(X)\le 2d$ bo'lishi ko'rsatilgan bo'lib, yuqoridagi teorema isbotsiz keltirilgan.

Maqolada parabolik Shteyn ko'pxilliklarda aniqlangan golomorf funksiyalar sinfi ρ – polinomlarning asosiy xossalari, xususan qamrov funksiyasining logarifmik singulyarliklari yordamida funksiyaning qutb tartiblari bilan bog'liqligi asoslab berilgan. Asosiy natija sifatida har qanday d – darajali polinomni chekli sondagi ratsional funksiyalar yig'indisi ko'rinishida ifodalash mumkinligi isbotlangan. Bu natija Loran yoyilmasi va Liuvill teoremasiga tayangan holda olingan bo'lib, parabolik ko'pxilliklarda polinomlar tuzilishini yanada chuqurroq tushunish imkonini beradi. Olingan natijalar kompleks analiz, ayniqsa ko'pxilliklar nazariyasida polinomlar va golomorf funksiyalarni o'rganishda muhim ahamiyatga ega bo'lib, keyingi tadqiqotlar uchun asos bo'lib xizmat qilishi mumkin.

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ELLIPSDAGI CHIZIQLAR OILASINING MODULI HAQIDA

Annotatsiya. *Chiziqlar oilasining moduli qaralayotgan sohalar oddiy(to'g'ri to'rtburchak, konsentrik halqa, fazoda parallelepiped kabi) bo'lgan hollardagina hosoblangan. Ushbu maqolada tekislikdagi ellipsni yon tomonlarini tutashtiruvchi chiziqlar oilasining moduli yordamchi konform akslantirishlar*