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REMAINDER TERMS FOR THE CRITICAL BELLMAN–HARRIS PROCESS ABOVE THE HEAVY-TAIL THRESHOLD

Annotation. *This paper investigates the second-order asymptotics of the survival probability for a critical Bellman–Harris process in the heavy-tail regime. The reproduction law is assumed to satisfy Slack's regular-variation condition, while the lifetime distribution has a regularly varying tail. Beyond the classical leading asymptotic law, the remainder term is determined by the interaction of two mechanisms: a renewal-driven mechanism associated with the equilibrium tail and a local mechanism represented by the Stone constant. A second threshold separates the resulting regimes, leading to a power correction, a constant contribution, or a logarithmic correction. The analysis is based on the two-term asymptotics of a single convolution arising from an exact identity for the survival probability. The results also recover the corresponding finite-variance expansion as a special case.*

Keywords: *Bellman-Harris process, critical branching process, regularly varying tails, heavy-tailed distribution, survival probability, remainder terms, second threshold, Stone constant, renewal density, equilibrium distribution, Slack's condition, slowly varying functions, second-order asymptotics.*

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OG'IR DUM CHEGARASIDAN YUQORIDA KRITIK BELLMAN-HARRIS JARAYONINING QOLDIQ HADLARI

Annotatsiya. *Ushbu maqolada kritik Bellman-Harris jarayonining og'ir dumli sharoitdagi omon qolish ehtimolining ikkinchi tartibli asimptotikasi o'rganiladi. Reproduktiv qonuniyat Slack sharti bo'yicha muntazam o'zgaruvchan, yashash muddati taqsimotining dumi esa muntazam o'zgaruvchan deb qaraladi. Klassik asimptotik qonuniyatdan keyingi qoldiq had ikkita asosiy mexanizmning – muvozanat dumi orqali yuzaga keluvchi yangilanish mexanizmi va Stone doimiysi bilan bog'liq lokal mexanizmning–o'zaro nisbatiga bog'liqligi ko'rsatiladi. Ikkinchi chegara asosida qoldiq hadning darajali, doimiy va logarifmik rejimlari ajratiladi. Ishda ushbu natijalar yagona konvolyutsiyaning ikki hadli asimptotikasi orqali asoslanadi hamda chekli dispersiyali holatda mavjud natijalar bilan mosligi ko'rsatiladi.*

Kalit so'zlar: *Bellman-Harris jarayoni, kritik tarmoqlanish jarayoni, muntazam o'zgaruvchan dumlar, og'ir dumli taqsimot, omon qolish ehtimoli, qoldiq hadlar, ikkinchi chegara, Stone doimiysi, yangilanish zichligi, muvozanat taqsimoti, Slack sharti, sekin o'zgaruvchi funksiyalar, ikkinchi tartibli asimptotika.*

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ОСТАТОЧНЫЕ ЧЛЕНЫ КРИТИЧЕСКОГО ПРОЦЕССА БЕЛЛМАНА-ХАРРИСА ВЫШЕ ПОРОГА ТЯЖЁЛОГО ХВОСТА

Аннотация. *В статье исследуется асимптотика второго порядка вероятности выживания критического процесса Беллмана–Харриса в режиме тяжёлых хвостов. Предполагается, что закон воспроизводства удовлетворяет условию регулярной вариации Слэка, а хвост распределения времени жизни является регулярно меняющимся. После классического главного асимптотического члена остаточный член определяется взаимодействием двух механизмов: механизма, обусловленного теорией возобновления и связанного с равновесным хвостом, и локального механизма, представленного константой Стоуна. Вторая пороговая граница разделяет соответствующие режимы и приводит к степенной, постоянной или логарифмической поправке. Анализ основан на двучленной асимптотике одной свёртки, возникающей из точного тождества для вероятности выживания. Полученные результаты согласуются с известным разложением в случае конечной дисперсии.*

Ключевые слова: *процесс Беллмана-Харриса, критический процесс ветвления, регулярно меняющиеся хвосты, тяжёлые хвосты, вероятность выживания, остаточные члены, вторая пороговая граница, константа Стоуна, плотность возобновления, равновесное распределение, условие Слэка, медленно меняющиеся функции, асимптотика второго порядка.*

1. Introduction. The Bellman-Harris process is the branching model in which a particle lives a random time τ with distribution function G and is replaced at the end of its life by a random number of offspring with probability generating function $f(s)=\sum_{k\geq 0}p_k s^k$, all lifetimes and offspring numbers

being independent. Writing $Z(t)$ for the number of particles alive at time $t \in \mathcal{T} := [0,\infty)$ and taking $Z(0) = 1$, the generating function $F(t;s) := \mathbb{E}s^{Z(t)}$ satisfies the non-linear renewal equation of Chover and Ney [3], and the standard exposition of the model is the monograph of Athreya and Ney [1]. The process is called critical when $f'(1) = 1$, which we assume throughout; criticality forces extinction with probability one, and the entire interest lies in the rate at which the survival probability

$$Q(t) := \mathbb{P}\{Z(t) > 0\} = 1 - F(t;0)$$

tends to zero.

The classical answer is a linear law. If the reproduction law has finite variance and the lifetime distribution has a finite second moment, then $Q(t) \sim 2\mu/(f''(1)t)$ as $t \to \infty$, where $\mu := \mathbb{E}\tau$; this is [3, Th~2], sharpened there to a three-term expansion with a logarithmic middle term. Neither hypothesis is an artefact of the method. On the lifetime side, Chover and Ney record that $t^2\overline{G}(t) \to 0$ suffices and that Kesten had shown it to be necessary, and on the reproduction side the finite variance was removed by Goldstein and Hoppe [6], who replaced it by the regular-variation condition of Slack [10], obtaining the corresponding non-linear law under moment conditions on the lifetime distribution.

Once the lifetime tail is heavy enough the linear law fails, and what replaces it has been known for a long time. Under Slack's condition of index ν and a lifetime tail regularly varying of index $-\beta$, Vatutin [12] proved that the survival probability is regularly varying of index $-\beta/(1+\nu)$, so that it decays like a power of the lifetime tail rather than like a reciprocal power of t; the relation behind that statement, namely the asymptotic equality of the reproduction defect $\varphi(Q)$ with the lifetime tail itself, was afterwards freed of regular variation by Yakymiv [13, Th~16]. The two regimes are separated by the threshold

$$\beta_* = \frac{1+\nu}{\nu},$$

below which that balance relation holds and above which the law of [6] persists. For $\nu = 1$ one has $\beta_* = 2$, in exact agreement with the necessity of Kesten's condition; for $\nu < 1$ the threshold moves upward, so that a heavier reproduction tail demands a lighter lifetime tail if the classical law is to survive at all.

What is left open by that body of work, on the far side of the threshold, is everything beyond the leading term: the law of [6] stops at a factor $1 + o(1)$, and nothing is said about what follows it. The present paper computes the second order there, and the answer is that a second threshold appears. The correction is decided by the comparison of two mechanisms, one renewal-driven and carried by the equilibrium tail of G, the other local and carried by the Stone constant of the renewal density, and their comparison is the sign of

$$\vartheta = \beta_{**} - \beta \qquad \text{with} \qquad \beta_{**} = \frac{1+2\nu}{\nu} = \beta_* + 1.$$

For $\vartheta > 0$ the correction is a power of order $t^{\vartheta}L(t)$; for $\vartheta < 0$ that power converges and the local mechanism takes over, leaving a logarithm; and at $\vartheta = 0$ the two fall on one scale and their coefficients add, both being negative, so that they reinforce one another. All three are modes of one and the same integral. The structural point is that β_{**}, like β_*, grows without bound as $\nu \downarrow 0$: a heavy reproduction tail demands a lighter lifetime tail not only for the leading law but separately for each of the three regimes of the remainder, and the finite-variance value $\beta_{**} = 3$ is the smallest that this second threshold can take.

The route is an exact identity. Writing $\Theta := \varphi(Q)$ for the reproduction defect evaluated along the process, which is non-increasing under no hypothesis at all, and v for the renewal-density defect in the terminology of [7], the survival probability satisfies

$$Q(t) = \frac{1}{\mu}\int_t^{\infty} \Theta(y)\mathbf{d}y - (v * \Theta)(t),$$

together with the mass identity $\int_0^\infty \Theta = \mu$; and the whole of the second-order structure is the two-term asymptotics of that single convolution, its two terms being the two mechanisms just named. The renewal input is the second-order theory of the renewal density of [7], which rests in turn on Stone [11] and on the theory of regular variation as presented in Bingham, Goldie and Teugels [2]. No estimate of Q' is used at any point, and none is available.

The paper is confined to the range $\beta > \beta_*$, and its relation to two neighbouring works should be stated at once. Below the threshold the balance relation of Vatutin and Yakymiv is the leading law, and its second- and third-order structure, together with the conditional limit laws and the behaviour at the threshold itself, are the subject of [8]; nothing of that range is treated here, and nothing proved there is reproved. Above the threshold the leading law is classical, and is recorded there as such. The finite-variance case $\nu = 1$ of the expansions below is the branching application of [7], where the range $2 \le \beta < 3$ is treated; that range is exactly (β_*, β_{**}) at $\nu = 1$, so that the present paper stands to it as the general index stands to the finite variance, and the agreement of the two is verified in Remark 2.

Section 2 states the hypotheses and the four principal results. Section 3 establishes the exact identity and the representation on which everything rests. Section 4 contains the proofs, and Section 5 illustrates the trichotomy numerically on a single lifetime distribution at three values of the reproduction index, one for each regime.

2. Hypotheses, Notation and Principal Results. Recall that a measurable function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying at infinity, written $L \in \mathcal{SV}_\infty$, if $L(cx)/L(x) \to 1$ as $x \to \infty$ for every fixed $c > 0$, and that $V \in \mathcal{RV}_\infty(a)$ means that $V(x) = x^a L(x)$ holds for some $L \in \mathcal{SV}_\infty$; the same notation at the origin is written $\mathcal{RV}_0(a)$. The standard reference for both is Bingham, Goldie and Teugels [2].

Hypothesis $(\mathcal{F})$. *The reproduction generating function is critical,* $f'(1) = 1$ *with* $p_1 < 1$, *and*

$$\varphi(u) := f(1-u) - (1-u) = u^{1+\nu}\ell\left(\frac{1}{u}\right) \qquad \text{for} \quad u \in (0,1], \tag{2.1}$$

where $\nu \in (0,1]$ *and* $\ell \in \mathcal{SV}_\infty$.

Hypothesis $(\mathcal{G})$. *The lifetime distribution is non-lattice with* $G(0+) = 0$, *admits a density* g, *and*

$$\overline{G}(t) = t^{-\beta} L(t) \qquad \text{for} \quad \beta > \beta_*, \tag{2.2}$$

where $L \in \mathcal{SV}_\infty$ *and* β_* *is the threshold (2.3).*

The two indices enter through three combinations only, namely the threshold, the balance parameter and the second balance point,

$$\beta_* := \frac{1+\nu}{\nu}, \qquad \beta_{**} := \frac{1+2\nu}{\nu} = \beta_* + 1 \qquad \text{and} \qquad \vartheta := \beta_{**} - \beta. \tag{2.3}$$

Since $\nu \le 1$ one has $\beta_* \ge 2$, so that (2.2) forces $\mu_2 < \infty$ and the Stone constant below exists with no hypothesis of its own; and $\beta_{**} \ge 3$, so that the third moment of G is finite only in the range $\vartheta < 0$.

We write $\mu := \mathbb{E}\tau$ and $\mu_2 := \mathbb{E}\tau^2$ for the first two moments of the lifetime, both finite under (2.2),

$$U := \sum_{n \ge 0} G^{*n}, \qquad u := U', \qquad v(t) := u(t) - \frac{1}{\mu} \qquad \text{and} \qquad \mathcal{S}_\mu := \frac{\mu_2}{2\mu^2} \tag{2.4}$$

for the renewal function, its density, the renewal-density defect and the Stone constant, the term $n = 0$ in the first sum being the unit point mass at the origin. The equilibrium tail of G is

$$\mathcal{Q}_G(t) := \frac{1}{\mu}\int_t^\infty \overline{G}(y)\,\mathrm{d}y, \qquad \text{sothat} \quad \int_0^\infty \mathcal{Q}_G(y)\,\mathrm{d}y = \mu\mathcal{S}_\mu. \tag{2.5}$$

Following [7] we call v the renewal-density defect and $\mathcal{S}_\mu$ the Stone constant. The object organising the analysis is not Q itself but the reproduction defect evaluated along the process,

$$\Theta(t) := \varphi(Q(t)), \tag{2.6}$$

which is non-increasing under no hypothesis at all, since $s \mapsto f(s) - s$ is strictly decreasing on $[0,1)$ while $t \mapsto 1 - Q(t)$ is non-decreasing. Alongside it we write

$$\Psi := \nu * \Theta \qquad \text{and} \qquad \Delta := \Psi - (\mathcal{S}_\mu - 1)\Theta - \mathcal{Q}_G \tag{2.7}$$

for the single convolution in which the whole second-order structure resides and for its residual. Finally, the norming function of the reproduction law is

$$\mathcal{N}(q) := \int_q^{q_0} \frac{\mathbf{d}r}{\varphi(r)}, \qquad \text{sothat} \quad \mathcal{N}(q) \sim \frac{1}{\nu q^\nu \ell(1/q)} \quad \text{as} \quad q \downarrow 0 \tag{2.8}$$

by Karamata's theorem, $q_0 \in (0,1)$ being fixed and a change of it altering $\mathcal{N}$ by a constant only. In this notation the classical law above the threshold reads $\mu\mathcal{N}(Q(t)) \sim t$ as $t \to \infty$, and every result below is a statement about the difference $\mu\mathcal{N}(Q(t)) - t$.

2.1. The dichotomy

Theorem 1. *Let Hypotheses $(\mathcal{F})$ and $(\mathcal{G})$ hold. Then, as $t \to \infty$,*

$$\mu\mathcal{N}(Q(t)) = t - \int_{t_0}^t \frac{\overline{G}(y)}{\Theta(y)}\mathbf{d}y(1+o(1)) + \mu(\mathcal{S}_\mu - 1)\log\Theta(t) + c + o(1), \tag{2.9}$$

the integrand being regularly varying of index $\vartheta - 1$ and $\log\Theta(t)$ being $-(1+1/\nu)\log t$ up to a term $o(\log t)$. The three regimes of (2.9) are the three modes of that single integral, exactly as the trichotomy of the linear theory is the trichotomy of one integral.

Theorem 2. *Let $\vartheta > 0$ and $\ell(x) \to \ell_\infty \in (0,\infty)$ as $x \to \infty$. Then the integral of (2.9) dominates the logarithm, and*

$$Q^\nu(t) = \frac{\mu}{\nu\ell_\infty t}\left[1 + \frac{\mathcal{C}_\nu}{\vartheta} t^{\vartheta-1}L(t) + o(t^{\vartheta-1}L(t))\right], \qquad \text{where} \quad \mathcal{C}_\nu := \frac{\nu}{\mu}\left(\frac{\nu\ell_\infty}{\mu}\right)^{1/\nu}. \tag{2.10}$$

The correction is positive: a heavy lifetime tail keeps particles in play longer, and the process survives better than the classical law predicts.

Theorem 3. *Let $\vartheta = 0$, that is $\beta = \beta_{**}$, and let $\ell(x) \to \ell_\infty \in (0,\infty)$ as $x \to \infty$. Then the integrand of (2.9) is regularly varying of index -1, and, as $t \to \infty$,*

$$\mu\mathcal{N}(Q(t)) = t - \mathcal{C}_\nu \Lambda_L(t) - \mu(\mathcal{S}_\mu - 1)(1 + \frac{1}{\nu})\log t + c + o(1 + \Lambda_L(t)), \qquad \Lambda_L(t) := \int_1^t \frac{L(y)}{y}\mathbf{d}y, \tag{2.11}$$

with $\mathcal{C}_\nu$ as in (2.10). Both coefficients are negative, so that the two mechanisms reinforce one another; and if moreover $L(t) \to L_\infty \in (0,\infty)$ they fall on one scale and combine into the single logarithmic term of coefficient

$$-\left[\mathcal{C}_\nu L_\infty + \mu(\mathcal{S}_\mu - 1)\frac{1+\nu}{\nu}\right]. \tag{2.12}$$

Remark 1. *Formula (2.11) admits four readings, and which of them applies is settled by L and by nothing else. If $\int_1^\infty [L(y)/y]\mathbf{d}y < \infty$, then Λ_L is bounded and is absorbed into the constant, the second-order term being the logarithm alone. If $\Lambda_L(t) \to \infty$ while $L(t) \to 0$, then $\Lambda_L(t) = o(\log t)$, the logarithm remains the second term and Λ_L is interposed before the constant, which therefore no longer exists as such. If $L(t) \to L_\infty \in (0,\infty)$, the two coefficients add as in (2.12), and this superposition is the genuine content of the balance point. If finally $L(t) \to \infty$, the renewal-driven term dominates the logarithm outright. Unlike the corresponding statement of the linear theory, the logarithm here is never absent, since it is carried by the Stone constant and $\mathcal{S}_\mu$ is finite throughout (2.2).*

Theorem 4. *Let $\vartheta < 0$. Then the integral of (2.9) converges and is absorbed into the constant, the logarithm becomes the second-order term, and*

$$\mu\mathcal{N}(Q(t)) = t - \mu(\mathcal{S}_\mu - 1)(1+\frac{1}{\nu})\log t + c + \mathcal{O}(t^{\vartheta}L(t)) \qquad \text{as} \quad t \to \infty. \tag{2.13}$$

Remark 2. *The finite-variance case is recovered in both regimes. For $\nu = 1$ the defect is $\varphi(u) = au^2 - bu^3 + \mathcal{O}(u^4)$ with $a = f''(1)/2$ and $b = f'''(1)/6$, so that the slowly varying factor ℓ is not constant and*

$$\mathcal{N}(q) = \frac{1}{aq} + \frac{b}{a^2}\log\frac{1}{q} + \text{const} \qquad \text{as} \quad q \downarrow 0. \tag{2.14}$$

Substituting (2.14) into (2.9) and inverting, the coefficient of $\log t$ in $1/Q(t)$ comes out as $2a - a\mu_2/\mu^2 - b/a$, which is exactly the coefficient $(a^2 - b)/a - a\sigma^2/\mu^2$ of the classical three-term expansion; and for $\vartheta > 0$ the power term of (2.10) reduces to $-a^2 t^{3-\beta}L(t)/(\mu^3(3-\beta))$, the second-order term of the remainder theory for the non-linear renewal equation at $p = -1$. The threshold of that theory, $\beta = 3$, is β_{**} at $\nu = 1$.

Remark 3. *The identification just made is the reason why (2.9) is stated in $\mathcal{N}$ and left uninverted. The third moment of the reproduction law enters the expansion not through the Stone mechanism, whose coefficient $\mu(\mathcal{S}_\mu - 1)$ depends on the lifetime distribution alone, but through the second term of $\mathcal{N}$, that is through the departure of ℓ from its limit. Whatever the reproduction law carries beyond that limit is carried by $\mathcal{N}$ itself, and an inversion performed under $\ell \to \ell_\infty$ discards it. That is harmless for $\vartheta > 0$, where the logarithm sits below the power of (2.10), and it is not harmless for $\vartheta \le 0$, where the logarithm is the second-order term; the statement (2.13) is therefore to be read with $\mathcal{N}$ computed from φ and not from its leading term.*

2.2. The constant term

Theorem 5. *Let the hypotheses of Theorem 4 hold, so that $\vartheta < 0$. Then the constant of (2.13) is*

$$c = \frac{\mu}{p_0} - \mu(\mathcal{S}_\mu - 1)\log p_0 - \int_0^\infty \frac{\overline{G}(y)}{\Theta(y)}\mathbf{d}y - \mu\int_0^\infty \Delta(y)\mathbf{d}(\frac{1}{\Theta})(y), \tag{2.15}$$

where $p_0 = f(0) = \Theta(0)$ and $\mathcal{N}$ is normalised by $\mathcal{N}(1) = 0$, the third integral converging because $\vartheta < 0$ and the fourth by Proposition 2.

Remark 4 *The identification (2.15) carries a single functional of the trajectory, against the two required by the corresponding statement of the linear theory. The one that disappears is the one attached to the logarithm: there the logarithm is produced against $\log t$ and the discrepancy between the trajectory and its leading term has to be integrated, whereas (2.9) produces it against $\log\Theta(t)$, where no integration is involved at all. What remains is the residual Δ of (2.7), and its convergence is not automatic: the integrator $\mathbf{d}(1/\Theta)$ carries a divergent logarithm of its own, so that $\Delta = o(\Theta)$ does not by itself suffice and a rate is needed.*

3. The Exact Identity and the Reduction. The whole of the second-order analysis rests on one identity, which holds with no hypothesis whatever and which reduces the entire question to the two-term asymptotics of a single convolution.

Lemma 1. *For every $t \ge 0$,*

$$Q(t) = \frac{1}{\mu}\int_t^\infty \Theta(y)\mathbf{d}y - (\nu * \Theta)(t), \qquad \text{and} \qquad \int_0^\infty \Theta(y)\mathbf{d}y = \mu. \tag{3.1}$$

Lemma 2. *With $\mathcal{N}$ normalised by $\mathcal{N}(1) = 0$, for every $t > 0$,*

$$\mu\mathcal{N}(Q(t)) = t + \frac{\mu\Psi(t)}{\Theta(t)} - \mu\int_0^t \Psi(y)\mathbf{d}(\frac{1}{\Theta})(y), \tag{3.2}$$

both terms being finite at the origin because $\Theta(0) = p_0 > 0$.

Every term of (2.9) is read off (3.2) by substituting the splitting $\Psi = (\mathcal{S}_\mu - 1)\Theta + \mathcal{Q}_G + \Delta$ of (2.7): the first summand produces the logarithm $\mu(\mathcal{S}_\mu - 1)\log\Theta(t)$, the second the power integral $\int \overline{G}/\Theta$ together with a boundary term $\mu\mathcal{Q}_G/\Theta$ which cancels exactly against the one carried by $\mu\Psi/\Theta$, and the third the residual functional of (2.15).

Remark 5. *The two mechanisms are best compared relatively to Q itself. Dividing (4.1) by Q and using the classical law in the form $\Theta \sim \mu Q/(\nu t)$, the renewal-driven mechanism contributes*

$$\frac{\mathcal{Q}_G(t)}{Q(t)} \in \mathcal{RV}_\infty(-(\beta - \beta_*)) \qquad \text{andthelocalone} \qquad (\mathcal{S}_\mu - 1)\frac{\Theta(t)}{Q(t)} \sim \frac{\mu(\mathcal{S}_\mu - 1)}{\nu t} \tag{3.3}$$

as $t \to \infty$. The first prevails when $\beta - \beta_* < 1$ and the second when $\beta - \beta_* > 1$, and that comparison is the sign of ϑ. The threshold β_* decides whether the classical law holds at all; the second threshold $\beta_{**} = \beta_* + 1$ decides which of the two corrections to it prevails, and the unit separating them is the order of the local mechanism, which is $1/t$ whatever the two indices may be.

The two mechanisms of the paper are the two terms of $\nu * \Theta$: the local one $\|\nu\|_1\Theta = (\mathcal{S}_\mu - 1)\Theta$ and the renewal-driven one $\|\Theta\|_1\nu = \mu\nu \sim \mathcal{Q}_G$. Which of the two a convolution lemma of the type of [7, Lemma 3] can deliver with its exact constant is decided by the sign of ϑ, and in each regime it delivers precisely the dominant one, the subordinate term falling below its own residual. This is structural and not fortuitous.

4. Proofs of the Principal Results. 4.1. The two identities. *Proof of Lemma 1.* Setting $s = 0$ in the basic equation and using (2.1) gives the renewal equation $Q = \overline{G} + Q * \mathbf{d}G - \Theta * \mathbf{d}G$, whence by successive substitution

$$Q = (\overline{G} - \Theta * \mathbf{d}G) * U = \overline{G} * U - \Theta * \mathbf{d}G * U.$$

Now $\overline{G} * U \equiv 1$ and $\mathbf{d}G * U = U - \delta_0$, so that $Q = 1 - \Theta * U + \Theta$. Since G admits a density, U splits into the unit point mass at the origin and an absolutely continuous part of density $1/\mu + \nu$, and

$$(\Theta * U)(t) = \Theta(t) + \frac{1}{\mu}\int_0^t \Theta(y)\mathbf{d}y + \Psi(t),$$

whence $Q(t) = 1 - \frac{1}{\mu}\int_0^t \Theta(y)\mathbf{d}y - \Psi(t)$. Both Θ and ν are absolutely integrable and vanish at infinity, so splitting the convolution at $t/2$ gives $\Psi(t) \to 0$; and $Q(t) \to 0$ by criticality. Letting $t \to \infty$ in the last display therefore leaves $\int_0^\infty \Theta(y)\mathbf{d}y = \mu$, and substituting that back turns the truncated integral into the tail integral, which is (3.1).

Proof of Lemma 2. The function Θ is monotone and hence of locally bounded variation, and ν is locally integrable, so $\Psi = \nu * \Theta$ is locally absolutely continuous. Differentiating (3.1) therefore gives $Q' = -\Theta/\mu - \Psi'$ almost everywhere, and dividing by Θ, which is strictly positive,

$$-\frac{\mu Q'(t)}{\Theta(t)} = 1 + \frac{\mu\Psi'(t)}{\Theta(t)}.$$

Since $\mathcal{N}' = -1/\varphi$ and $\Theta = \varphi(Q)$, the left-hand side is $\mu\mathbf{d}\mathcal{N}(Q(t))/\mathbf{d}t$. Integrating from the origin, where $\mathcal{N}(Q(0)) = \mathcal{N}(1) = 0$, and integrating the remaining term by parts against $1/\Theta$, which is finite at the origin because $\Theta(0) = p_0 > 0$, gives (3.2).

4.2. The two-term asymptotics of the convolution

Everything now rests on Ψ, and on nothing else. We isolate the two inputs required.

Hypothesis $(\mathcal{B})$. *There is a constant* C *with* $|v(t)| \le C\mathcal{Q}_G(t)$ *for all large* t, *and, for every fixed* T,

$$U(t)-U(t-y)-\frac{y}{\mu} \sim \frac{y\mathcal{Q}_G(t)}{\mu} \qquad \text{as} \quad t\to\infty, \quad \text{uniformlyfor} \quad y\in[0,T].$$

Proposition 1 *Under Hypothesis [CBAhypB]* $(\mathcal{B})$, *as* $t\to\infty$,

$$\Psi(t) = (\mathcal{S}_\mu - 1)\Theta(t) + \mathcal{Q}_G(t)(1+o(1)) + o(\Theta(t)). \tag{4.1}$$

Proof. Write $\sigma := -\mathbf{d}\Theta$, a positive measure of total mass $\Theta(0) = p_0$ and, by the mass identity of (3.1), of first moment $\int_{[0,\infty)} y\sigma(\mathbf{d}y) = \int_0^\infty \Theta(y)\mathbf{d}y = \mu$. With $W(t) := U(t) - t/\mu - \mathcal{S}_\mu$, so that $W(t)-W(t-y) = \int_{t-y}^t v(r)\mathbf{d}r$, Fubini's theorem turns the convolution into

$$\Psi(t) = (\mathcal{S}_\mu - 1)\Theta(t) + W(t)\Theta(t) + E(t), \qquad E(t) := \int_{[0,t)} [W(t)-W(t-y)]\sigma(\mathbf{d}y).$$

The middle term is negligible: $|W(t)| = \mathcal{O}(t\mathcal{Q}_G(t))$ by the first half of Hypothesis [CBAhypB] $(\mathcal{B})$, and $t\Theta(t)\to 0$, so that $W\Theta = o(\mathcal{Q}_G)$. It remains to treat E, and we split the range of y into four blocks, with T fixed, $\varepsilon\in(0,1/2)$ fixed and the last cut at $t-\sqrt{t}$.

On $[0,T]$ the second half of the hypothesis applies and gives $W(t)-W(t-y)\sim y\mathcal{Q}_G(t)/\mu$ uniformly, so that this block contributes $\mathcal{Q}_G(t)\mu^{-1}\int_{[0,T]} y\sigma(\mathbf{d}y)$ up to a factor $1+o(1)$, and the integral tends to μ as $T\to\infty$ because the first moment of σ is finite. On $[T,\varepsilon t]$ the bound $|W(t)-W(t-y)| \le Cy\mathcal{Q}_G(t-y)/\mu \le C' y\mathcal{Q}_G(t)$, valid for $y\le \varepsilon t$ by the first half of the hypothesis and the regular variation of $\mathcal{Q}_G$, gives a contribution at most $C'\mathcal{Q}_G(t)\int_{[T,\infty)} y\sigma(\mathbf{d}y)$, and the last integral tends to zero as $T\to\infty$. On $[\varepsilon t, t-\sqrt{t}]$ both arguments of W exceed $\sqrt{t}$, so the increment is at most $2\sup_{r\ge\sqrt{t}}|W(r)|\to 0$, while the σ -mass of that block is at most $\Theta(\varepsilon t) = \mathcal{O}(\Theta(t))$ by regular variation; the block is therefore $o(\Theta(t))$. On $[t-\sqrt{t},t)$ the increment is bounded by $2\sup_{r\ge 0}|W(r)|$, which is finite, while the σ -mass is $\Theta(t-\sqrt{t})-\Theta(t) = o(\Theta(t))$, again because Θ is regularly varying and $\sqrt{t} = o(t)$. Letting $t\to\infty$ and then $T\to\infty$ collects the four blocks into (4.1).

Remark 6. *Hypothesis* $(\mathcal{B})$ *is what replaces the Tauberian side condition of the linear theory. Both of its halves are statements about increments of the renewal function, and neither requires anything of the monotonicity of its density, so that it is strictly weaker than ultimate monotonicity of* v *together with the monotone density theorem. We have not been able to discharge it from the literature, and the reason is structural rather than accidental. The sharp local theory of the remainder in renewal theory has been developed precisely where the second moment of the lifetime fails: the integrated statement is classical for finite mean and infinite second moment, where by a theorem of Sgibnev regular variation is not even needed, and the exact third-order behaviour was obtained by Doney [4] for tail index in* $(1,2)$, *with local versions in [5] and, for the density, in Isozaki [9]. Above the threshold one has* $\beta > \beta_* \ge 2$, *so that* $\mu_2 < \infty$ *throughout, and that is exactly the range in which those results do not speak. The hypothesis is therefore carried, and the question of removing it is a question about renewal theory and not about branching processes.*

For the constant term a rate is needed where (4.1) needs only an order.

Hypothesis $(\mathcal{B}')$. *Hypothesis* $(\mathcal{B})$ *holds with a rate: there are* $\kappa > 0$ *and* $\delta > 0$ *such that*

$$U(t)-U(t-y)-\frac{y}{\mu} = \frac{y\mathcal{Q}_G(t)}{\mu}[1+\mathcal{O}(t^{-\delta})] \qquad \text{as} \quad t\to\infty, \quad \text{uniformlyfor} \quad y\in[0,t^\kappa].$$

Proposition 2. *Under Hypothesis [CBAhypBq]*$(\mathcal{B}')$ *and* $\vartheta < 0$ *there is* $\delta' > 0$ *with* $\Delta(t) = \mathcal{O}(\Theta(t)t^{-\delta'})$ *as* $t \to \infty$, *and consequently the integral* $\int_0^\infty \Delta \mathbf{d}(1/\Theta)$ *converges.*

Proof. Run the four blocks of the proof of Proposition 1 with $T = t^\kappa$. The first block then carries the relative error $\mathcal{O}(t^{-\delta})$ of Hypothesis [CBAhypBq]$(\mathcal{B}')$. The second is bounded by $C'\mathcal{Q}_G(t)\int_{[T,\infty)} y\sigma(\mathbf{d}y)$, and the tail of the first moment of σ is $T\Theta(T) + \int_T^\infty \Theta$, a quantity regularly varying of index $-1/\nu$, so that this block is $\mathcal{O}(\mathcal{Q}_G(t)t^{-\kappa/\nu})$. The third is at most $2\Theta(\varepsilon t)\sup_{r \geq \sqrt{t}} |W(r)|$, and $|W(r)| = \mathcal{O}(r\mathcal{Q}_G(r))$ is regularly varying of index $2 - \beta < 0$, so that this block is $\mathcal{O}(\Theta(t)t^{(2-\beta)/2})$. The fourth is bounded by $\Theta(t - \sqrt{t}) - \Theta(t)$ times a constant, and that difference is $\mathcal{O}(\Theta(t)t^{-1/2})$ because Θ is regularly varying. Every block therefore carries a power rate, and since $\mathcal{Q}_G(t) = \mathcal{O}(\Theta(t)t^{\vartheta})$ for $\vartheta < 0$, all of them are $\mathcal{O}(\Theta(t)t^{-\delta'})$ with δ' the smallest of the exponents. Finally $\mathbf{d}(1/\Theta) = \sigma/\Theta^2$ and $\sigma/\Theta \asymp (1 + 1/\nu)\mathbf{d}y/y$ by regular variation, so the integrand of the functional is $\mathcal{O}(y^{-1-\delta'})$ and the integral converges at infinity; at the origin it is bounded.

4.3. Proofs of the four theorems.

Proof of Theorem 1. Substitute the splitting $\Psi = (\mathcal{S}_\mu - 1)\Theta + \mathcal{Q}_G + \Delta$ of (2.7) into the representation (3.2). The first summand contributes

$$\mu(\mathcal{S}_\mu - 1) - \mu(\mathcal{S}_\mu - 1)\int_0^t \Theta(y)\mathbf{d}\left(\frac{1}{\Theta}\right)(y) = \mu(\mathcal{S}_\mu - 1) + \mu(\mathcal{S}_\mu - 1)\log\frac{\Theta(t)}{p_0},$$

which is the logarithmic term of (2.9) together with a constant. The second contributes, on integrating by parts back and using $\mathcal{Q}_G' = -\overline{G}/\mu$,

$$\frac{\mu\mathcal{Q}_G(t)}{\Theta(t)} - \mu\left[\frac{\mathcal{Q}_G}{\Theta}\right]_0^t - \int_0^t \frac{\overline{G}(y)}{\Theta(y)}\mathbf{d}y = \frac{\mu}{p_0} - \int_0^t \frac{\overline{G}(y)}{\Theta(y)}\mathbf{d}y,$$

the two boundary terms at t cancelling exactly and $\mathcal{Q}_G(0) = 1$ fixing the one at the origin. The third contributes the residual functional. Finally the integrand $\overline{G}/\Theta$ is regularly varying of index $\vartheta - 1$, since $\Theta \in \mathcal{RV}_\infty(-(1 + 1/\nu))$ by the classical law, and $\log\Theta(t) = -(1 + 1/\nu)\log t + o(\log t)$ because the slowly varying factor of Θ has logarithm $o(\log t)$. This is (2.9).

Proof of Theorem 2. Let $\vartheta > 0$. Karamata's theorem applied to the integrand of index $\vartheta - 1$ gives $\int_{t_0}^t \overline{G}/\Theta \sim \mathcal{C}_\nu t^\vartheta L(t)/\vartheta$, with $\mathcal{C}_\nu$ as in (2.10) once $\ell(x) \to \ell_\infty$ is used to evaluate Θ; this dominates both the logarithm and the constant. Since $\mathcal{N}(q) \sim 1/(\nu q^\nu \ell_\infty)$, inverting (2.9) gives (2.10). The sign is that of the subtracted integral, hence the correction to Q^ν is positive.

Proof of Theorem 3. At $\vartheta = 0$ the integrand of (2.9) is regularly varying of index -1 and equals $\mathcal{C}_\nu L(y)/y$ up to a factor $1 + o(1)$, so its integral is $\mathcal{C}_\nu \Lambda_L(t)$ to the same accuracy. Adding the logarithmic term of (2.9) gives (2.11), and $\Lambda_L(t) \sim L_\infty \log t$ under the further convergence of L gives (2.12).

Proof of Theorems 4. For $\vartheta < 0$ the integrand of (2.9) is regularly varying of index $\vartheta - 1 < -1$ and is bounded at the origin, so the integral converges and its tail is $\mathcal{O}(t^\vartheta L(t))$ by Karamata's theorem. The logarithm is then the second-order term, which is (2.13). Collecting the constants produced in the proof of Theorem 1, namely μ/p_0 from the equilibrium tail and $-\mu(\mathcal{S}_\mu - 1)\log p_0$ from the logarithm, together with the limit of the convergent integral and the residual functional, gives (2.15).

Conclusion. In this paper, we have studied the second-order asymptotic behaviour of the survival probability of a critical Bellman-Harris process above the heavy-tail threshold. Under Slack's regular-variation condition for the reproduction law and regular variation of the lifetime tail, we have described the structure of the remainder beyond the classical leading asymptotic term.The analysis shows that the second-order behaviour is determined by two distinct mechanisms. The first mechanism is renewal-driven and is associated with the equilibrium tail of the lifetime distribution. The second is local and is represented by the Stone constant of the renewal density. The comparison of these two mechanisms gives rise to a second threshold, which determines the dominant form of the correction term.Three principal regimes are obtained. In one regime, the renewal-driven contribution produces a power-type correction. In another, the corresponding integral is absorbed into the constant and the logarithmic term becomes the leading second-order contribution. At the balance point, the two mechanisms occur on the same scale and their contributions combine, producing a logarithmic correction with the coefficients of both mechanisms. Thus, the logarithmic term remains an essential component of the second-order structure throughout the considered range.A central feature of the approach is that the entire second-order analysis is reduced to the two-term asymptotics of a single convolution obtained from an exact identity for the survival probability. This provides a unified framework for describing the different asymptotic regimes and clarifies the roles of the renewal-driven and local mechanisms.

Finally, the results are consistent with the known finite-variance case. In the appropriate special regime, the obtained expansion reduces to the previously established remainder expansion for the nonlinear renewal equation. Therefore, the results extend the known finite-variance theory to the more general setting of regularly varying reproduction laws and heavy-tailed lifetime distributions.

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DEPENDENCE OF THE ELECTRICAL CONDUCTIVITY AND TENZORESISTIVE PROPERTIES OF POLYCRYSTALLINE PbSe FILMS ON THE TECHNOLOGICAL PARAMETERS

Annotation. *The influence of technological parameters on the electrical conductivity and strain-sensitive properties of polycrystalline PbSe films was investigated. The effects of substrate temperature, deposition position, and film thickness on charge transport were analyzed. Changes in these parameters were found to affect electrical conductivity and may cause a transition from p-type to n-type conductivity. This behavior is attributed to compositional inhomogeneity, grain-boundary barriers, and charge-carrier redistribution. Optimal technological conditions provide enhanced electrical conductivity and strain sensitivity of PbSe films.*